Appendix D

ANSWERS TO SOME PROBLEMS

1-1. **(a)** At standard temperature and pressure, a mole of an ideal gas contains 6.022×10^{23} molecules (Avogadro's number) and occupies 22.4 liters. Hence, the number per m^3 is $6.022 \times 10^{23}/2.24 \times 10^{-2} = 2.66 \times 10^{25}\ m^{-3}$.

(b) Since $PV = NRT$, $n = N/V = P/RT$. Hence $n_1/n_0 = P_1T_0/P_0T_1$. Taking n_0 to be the density in part (a) and n_1 to be that in part (b), we have

$$n_1 = (2.69 \times 10^{25}) \frac{10^{-3}}{760} \frac{273}{(273 + 20)} = 3.30 \times 10^{19}\ m^{-3}$$

Note that a diatomic gas such as H_2 will have twice as many *atoms* per torr as, say, He.

1-2. Consider the integral

$$I^2 \equiv \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy = \iint_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$$

in a two-dimensional space. Transforming to cylindrical coordinates, we have

$$I^2 = \iint e^{-r^2} r\, dr\, d\phi = 2\pi \int_0^{\infty} e^{-r^2} r\, dr$$
$$= \pi \int e^{-r^2}\, d(r^2) = -\pi\, e^{-r^2}\big|_0^{\infty} = \pi$$

Hence,

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

and

$$1 = \int_{-\infty}^{\infty} \hat{f}(u)\,du = A\left(\frac{2KT}{m}\right)^{1/2} \int_{-\infty}^{\infty} e^{-mu^2/2KT}\, d\left[u\left(\frac{m}{2KT}\right)^{1/2}\right]$$

$$= AI\left(\frac{2KT}{m}\right)^{1/2}$$

$$\therefore\ A = (m/2\pi KT)^{1/2}$$

1-4.

$$p = n(KT_e + KT_i) = 10^{21}(4\times 10^4)(1.6\times 10^{-19})$$

$$= 6.4 \times 10^6\ \mathrm{N/m^2}$$

$$1\ \mathrm{atm} \simeq 10^5\ \mathrm{N/m^2} \qquad p = 64\ \mathrm{atm}$$

$$1\ \mathrm{atm} \simeq 14.7\ \mathrm{lb/in^2}. = (14.7)(144)/(2000)$$

$$= 1.06\ \mathrm{tons/ft^2}$$

$$p \simeq 68\ \mathrm{tons/ft^2}$$

1-5.

$$\frac{d^2\phi}{dx^2} = -\frac{e(n_i - n_e)}{\epsilon_0} = -\frac{1}{\epsilon_0} n_\infty e\left(e^{-e\phi/KT_i} - e^{e\phi/KT_e}\right)$$

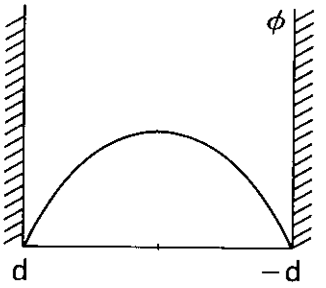
$$\simeq \frac{n_\infty e}{\epsilon_0}\left(\frac{e\phi}{KT_i} + \frac{e\phi}{KT_e}\right)$$

$$\phi = \phi_0\, e^{-|x|/\lambda_D}, \quad \text{where } \frac{1}{\lambda_D^2} = \frac{n_\infty e^2}{\epsilon_0}\left(\frac{1}{KT_e} + \frac{1}{KT_i}\right)$$

$$\text{If } T_i \ll T_e \qquad \lambda_D \simeq (KT_i\epsilon_0/n_\infty e^2)^{1/2}$$

$$\text{If } T_e \ll T_i \qquad \lambda_D \simeq (KT_e\epsilon_0/n_\infty e^2)^{1/2}$$

1-6. (a)



$$\frac{d^2\phi}{dx^2} = -\frac{nq}{\epsilon_0}$$

Let $\phi = Ax^2 + Bx + C$; $\phi' = 2Ax + B$; $\phi'' = 2A$. At $x = 0$, $\phi' = 0$ by symmetry $\therefore\ B = 0$. At $x = \pm d$, $\phi = 0$; therefore, $0 = Ad^2 + C$ and $C = -Ad^2$. Since

$$\phi'' = 2A = -\frac{nq}{\epsilon_0} \qquad A = -\frac{1}{2\epsilon_0} nq$$

and

$$\phi = Ax^2 - Ad^2 = \frac{1}{2\epsilon_0} nq(d^2 - x^2)$$

(b) Energy to move a charge q from x_1 to x_2 is change in potential energy $\Delta(q\phi) = q(\phi_2 - \phi_1)$. Let $\phi_1 = 0$ at $x = \pm d$ and $\phi_2 = (1/2\epsilon_0)nqd^2$ at $x = 0$. Then

$$\mathscr{E} = \frac{1}{2\epsilon_0} nq^2 d^2$$

Let $d = \lambda_D$; then

$$\mathscr{E} = \frac{1}{2\epsilon_0} nq^2 \frac{KT\epsilon_0}{nq^2} = \frac{1}{2} KT = E_{AV}$$

for a one-dimensional Maxwellian distribution. Hence, if $d > \lambda_D$, $\mathscr{E} > E_{AV}$. If the velocities are distributed in three dimensions, we have $E_{AV} = \frac{3}{2}KT$ and $\mathscr{E} > \frac{1}{3}E_{AV}$. The factor 3 is not important here. The point is that a thermal particle would not have enough energy to go very far in a plasma ($d \gg \lambda_D$) if the charge of one species is not neutralized by another species.

1-7. **(a)** $\lambda_D = 7400(2/10^{16})^{1/2} = 10^{-4}$ m, $N_D = 4.8 \times 10^4$.

(b) $\lambda_D = 7400(0.1/10^{12})^{1/2} = 2.3 \times 10^{-3}$ m, $N_D = 5.4 \times 10^4$.

(c) $\lambda_D = 7400(800/10^{23})^{1/2} = 6.6 \times 10^{-7}$ m, $N_D = 1.2 \times 10^5$.

2-1. $E = \frac{1}{2}mv_\perp^2 \;\therefore\; v_\perp = (2E/m)^{1/2}$, $r_L = mv_\perp/eB$.

(a)

$$v_\perp = \left[\frac{(2)(10^4)(1.6 \times 10^{-19})}{9.11 \times 10^{-31}}\right]^{1/2} = 5.93 \times 10^7 \text{ m/sec}$$

$$r_L = \frac{(9.11 \times 10^{-31})(5.93 \times 10^7)}{(1.6 \times 10^{-19})(0.5 \times 10^{-4})} = 6.75 \text{ m}$$

(b)

$$v_\perp = (300)(1000) = 3 \times 10^5 \text{ m/sec}$$

$$r_L = \frac{(1.67 \times 10^{-27})(3 \times 10^5)}{(1.6 \times 10^{-19})(5 \times 10^{-9})} = 6.26 \times 10^5 \text{ m} = 626 \text{ km}$$

(c)

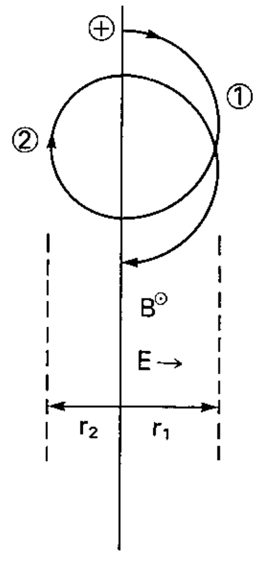
$$v_\perp = \left[\frac{(2)(10^3)(1.6 \times 10^{-19})}{(4)(1.67 \times 10^{-27})}\right]^{1/2} = 2.19 \times 10^5 \text{ m/sec}$$

$$r_L = \frac{(4)(1.67 \times 10^{-27})(2.19 \times 10^5)}{(1.6 \times 10^{-19})(5.00 \times 10^{-2})} = 0.183 \text{ m}$$

(d)

$$r_L = \frac{2ME}{qB} = \frac{[(2)(4)(1.67 \times 10^{-27})(3.5 \times 10^6)(1.6 \times 10^{-19})]^{1/2}}{(2)(1.6 \times 10^{-19})(8)}$$

$$= 3.38 \times 10^{-2} \text{ m}$$



2-4. Let initial energy be $\mathscr{E}_0$, and Larmor radii r_1 and r_2, as shown. Energy at ① is $\mathscr{E}_1 = \mathscr{E}_0 + eEr_1$; energy at ② is $\mathscr{E}_2 = \mathscr{E}_0 - eEr_2$. (It would be acceptable to say: $\mathscr{E}_{1,2} = \mathscr{E}_0 \pm eE\bar{r}_L$ here.) Also $v_{\perp 1,2}^2 = 2\mathscr{E}_{1,2}/M$. We are asked to make the approximation

$$r_{1,2} = \frac{Mv_{\perp 1,2}}{eB} = \frac{M}{eB}\left(\frac{2\mathscr{E}_{1,2}}{M}\right)^{1/2}$$
$$= \frac{1}{\Omega_c}\left(\frac{2\mathscr{E}_0}{M}\right)^{1/2}\left(1 + \frac{eE}{\mathscr{E}_0}r_{1,2}\right)^{1/2}$$

For small E, expand the square root in a Taylor series:

$$r_{1,2} \simeq \frac{1}{\Omega_c}\left(\frac{2\mathscr{E}_0}{M}\right)^{1/2}\left(1 \pm \frac{1}{2}\frac{eE}{\mathscr{E}_0}r_{1,2}\right)$$
$$r_{1,2} = \frac{1}{\Omega_c}\left(\frac{2\mathscr{E}_0}{M}\right)^{1/2}\left[1 \pm \frac{1}{2}\frac{eE}{\mathscr{E}_0}\frac{1}{\Omega_c}\left(\frac{2\mathscr{E}_0}{M}\right)^{1/2}\right]^{-1}$$
$$\simeq \frac{1}{\Omega_c}\left(\frac{2\mathscr{E}_0}{M}\right)^{1/2}\left[1 \pm \frac{1}{2}\frac{eE}{\mathscr{E}_0}\frac{1}{\Omega_c}\left(\frac{2\mathscr{E}_0}{M}\right)^{1/2}\right]$$

Thus

$$r_1 - r_2 = \frac{eE}{\mathscr{E}_0}\frac{1}{\Omega_c^2}\left(\frac{2\mathscr{E}_0}{M}\right) = \frac{2eE}{M\Omega_c^2}$$

independent of $\mathscr{E}_0$. The guiding center moves a distance $2(r_1 - r_2)$ in a time $2\pi/\Omega_c$, so

$$v_{gc} = 2(r_1 - r_2)(\Omega_c/2\pi) = \frac{4eE}{M\Omega_c}\frac{1}{2\pi} = \frac{2}{\pi}\frac{E}{B} \simeq \frac{E}{B}$$

Thus the guiding center drift is independent of the ion energy $\mathscr{E}_0$. The factor $2/\pi$ would be 1 if we did not make the crude approximation.

2-5. (a)

$$n = n_0 e^{e\phi/KT_e} \quad \therefore \quad \phi = (KT_e/e)\ln(n/n_0)$$
$$\mathbf{E} = -\frac{\partial\phi}{\partial r}\hat{\mathbf{r}} = -\frac{KT_e}{e}\frac{1}{n}\frac{\partial n}{\partial r}\hat{\mathbf{r}} = \frac{KT_e}{e\lambda}\hat{\mathbf{r}}$$

(b)

$$\mathbf{v}_E = -\frac{E_r}{B}\hat{\boldsymbol{\theta}} = -\frac{KT_e}{eB\lambda}\hat{\boldsymbol{\theta}}$$

Consider electrons:

$$v_{th} = \left(\frac{2KT_e}{m}\right)^{1/2} \quad \therefore \quad |v_E| = \frac{KT_e}{m}\frac{m}{eB}\frac{1}{\lambda} = \frac{1}{2}\frac{v_{th}^2}{\omega_c}\frac{1}{\lambda}$$

Now, $r_L = mv_\perp/eB$, so for a distribution of velocities we must find an average r_L. Since $v_\perp$ contains two degrees of freedom, we have

$$\tfrac{1}{2}m\overline{v_\perp^2} = 2 \times \tfrac{1}{2}KT_e$$

The most convenient average is

$$\langle v_\perp \rangle_{\text{rms}} = (2KT_e/m)^{1/2} = v_{\text{th}}$$

Using this for $v_\perp$ in r_L, we have

$$|v_E| = \frac{1}{2}\frac{v_{\text{th}}}{\lambda}\frac{v_\perp}{\omega_c} = \frac{1}{2}\frac{v_{\text{th}} r_L}{\lambda}$$

so that $|v_E| = v_{\text{th}}$ implies $r_L = 2\lambda$.

(c) If we take ions instead of electrons, we have $v_{\text{th}i} = (2KT_i/M)^{1/2} = v_{\perp i}$, $r_{Li} = v_{\perp i}/\omega_{ci}$, and

$$|v_E| = \frac{1}{2\lambda}\left(\frac{2KT_e}{M}\right)\left(\frac{M}{eB}\right) = \frac{1}{2\lambda}\frac{T_e}{T_i}\frac{v_{\text{th}i}v_{\perp i}}{\omega_{ci}} = \frac{1}{2}\frac{T_e}{T_i}v_{\text{th}i}r_{Li}$$

If $|v_E| = v_{\text{th}i}$, it is still true that $r_{Li} = 2\lambda$ *provided* that $T_i = T_e$.

2-6. (a)

$$n = n_0 \exp\left(e^{-r^2/a^2} - 1\right) = n_0 e^{e\phi/KT_e}$$

$$\therefore \quad \frac{e}{KT_e}\phi(r) = e^{-r^2/a^2} - 1$$

$$\mathbf{E} = -\nabla\phi = \frac{\partial\phi}{\partial r}\hat{\mathbf{r}} \qquad E_r(r) = -\frac{\partial\phi}{\partial r} = \frac{KT_e}{e}\frac{2r}{a^2}e^{-r^2/a^2}$$

$$\frac{dE_r}{dr} = \frac{2KT_e}{ea^2}\left(1 - \frac{2r^2}{a^2}\right)e^{-r^2/a^2} = 0 \qquad \frac{r^2}{a^2} = \frac{1}{2}$$

$$E_{\max} = \frac{KT_e}{ea}\frac{2}{\sqrt{2}}e^{-1/2} = \frac{(0.2)(1.6\times10^{-19})}{(1.6\times10^{-19})(.01)}\sqrt{2}\,e^{-1/2} = 17\,\frac{N}{C}$$

$$= 1700 \text{ V/m}$$

$$\mathbf{v}_E = -\frac{E_r}{B}\hat{\boldsymbol{\theta}} \qquad V_{E\max} = \frac{E_{\max}}{B} = \frac{17}{0.2} = 8500 \text{ m/sec}$$

(b) Compare the force Mg with the force eE for an ion. (mg for an electron would be 1836 times smaller.) $g = 9.80$ m/sec^2. $Mg = (39)(1.67\times10^{-27})(9.80) = 6.38\times10^{-25}$ N. $eE_{\max} = (1.6\times10^{-19})(17) = 2.75\times10^{-18}$ N $= 4\times10^6$ Mg. Hence gravitational drift 4 million times smaller.

(c)

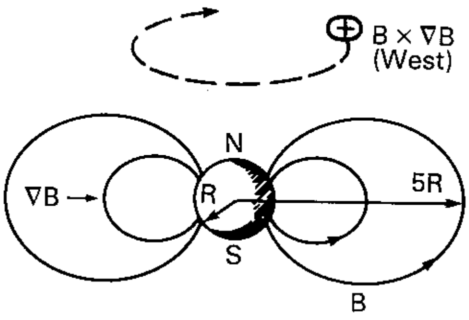
$$r_L = \frac{Mv_\perp}{eB} = 10^{-2}\ \mathrm{m}$$

$$v_\perp = (2KT/M)^{1/2} = \left[\frac{(2)(0.2)(1.6\times10^{-19})}{(39)(1.67\times10^{-27})}\right]^{1/2}$$

$$= 9.9\times10^{2}\ \mathrm{m/sec}$$

$$B = \frac{(39)(1.67\times10^{-27})(9.9\times10^{2})}{(10^{-2})(1.6\times10^{-19})} = 4.00\times10^{-2}\ \mathrm{T}$$

2-8.



$$B = \frac{c}{r^3} = \frac{0.3\times10^{-4}}{(r/R)^3}\ \mathrm{T}$$

$$v_{\nabla B} = \frac{1}{2}v_\perp r_L\left|\frac{\mathbf{B}\times\nabla\mathbf{B}}{B^2}\right| = \frac{1}{2}vr_L\left|\frac{\nabla B}{B}\right|$$

(a)

$$\nabla B = \frac{\partial B}{\partial r}\hat{\mathbf{r}} = -3\frac{c}{r^4}\hat{\mathbf{r}} = \frac{3}{r}B(-\hat{\mathbf{r}}) \qquad \left|\frac{\nabla B}{B}\right| = \frac{3}{r}$$

$$\frac{1}{2}v_\perp r_L = \frac{1}{2}\frac{v_\perp^2}{\omega_c} = \frac{1}{2}\frac{2KT/m}{eB/m} = \frac{KT}{eB}$$

$$= \frac{(1.6\times10^{-19})(KT)_{\mathrm{eV}}}{1.6\times10^{-19}}\frac{1}{B} = \frac{(KT)_{\mathrm{eV}}}{B}$$

$$B(r=5R) = \frac{0.3\times10^{-4}}{5^3} = 2.4\times10^{-7}\ \mathrm{T}$$

$$5R = (5)(4000\ \mathrm{mile})(1.6\ \mathrm{km/mile})(10^3\ \mathrm{m/km}) = 3.2\times10^{7}\ \mathrm{m}$$

$$v_{\nabla B} = 10^{8}\frac{(KT)_{\mathrm{eV}}}{2.4\times10^{-7}} = 0.39(KT)_{\mathrm{eV}}\ \mathrm{m/sec}$$

$$\text{Ions: } KT = 1\ \mathrm{eV} \qquad v_{\nabla B} = \underline{0.39\ \mathrm{m/sec}}$$

$$\text{Electrons: } KT = 3\times10^{4}\ \mathrm{eV} \qquad v_{\nabla B} = \underline{1.17\times10^{4}\ \mathrm{m/sec}}$$

(b) Ions: westward; electrons: eastward.

(c) $2\pi r = (6.28)(3.2\times10^7) = 2.0\times10^8$ m

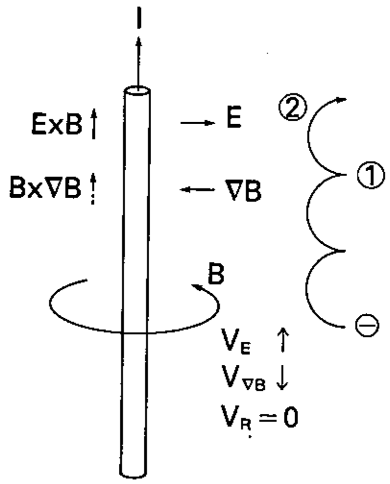
$$t = \frac{2\pi r}{v_{\nabla B}} = \frac{(2.0\times10^{8})}{(1.17\times10^{4})} = 1.7\times10^{4}\ \mathrm{sec} = 4.8\ \mathrm{hr}$$

(d)

$$j = nev_{\nabla B} \qquad \text{neglect ions}$$

$$= (10^7)(1.6\times10^{-19})(1.17\times10^{4}) = 1.87\times10^{-8}\ \mathrm{A/m^2}$$

2-9. (a) $v_R = 0$, since the electron gains no energy in the parallel ($\hat{\theta}$) direction. Since the electron starts at rest with no thermal energy, it will come back to rest after one cycle. Hence, the orbit has sharp cusps instead of loops. It is clear that the v_E drift must dominate, since the electron starts to the left, and the Lorentz force makes it move upwards.



(b) In cylindrical geometry, $\phi = A \ln r + B$. Since

$$\phi(10^{-3}) = 460 \text{ V} \quad \text{and} \quad \phi(0.1 \text{ m}) = 0,$$

$$460 = A \ln(10^{-3}) + B$$

$$0 = A \ln(0.1) + B \qquad B = -A \ln(0.1)$$

$$460 = A \ln(10^{-3}) - A \ln(0.1)$$

$$= A \ln(0.01) \qquad A = 460/\ln(0.01)$$

$$\phi(r) = \frac{460}{\ln(0.01)}[\ln r - \ln(0.1)] = 460\frac{\ln(0.1r)}{\ln 100} V$$

$$E = \frac{-\partial\phi}{\partial r} = \frac{-460}{\ln 100}\left(\frac{r}{0.1}\right)\left(\frac{-0.1}{r^2}\right) = \frac{460/r}{\ln 100}\frac{V}{m}$$

$$= \frac{460}{(4.6)(1)} = 10^4 \frac{V}{m} \text{ at } r = 10^{-2} \text{ m}$$

$$B = \frac{I(A)10^{-4}}{5r} = \frac{500 \times 10^{-4}}{(5)(1)} = 0.01 \text{ T}$$

$$|v_E| = |E/B| = 10^8 \frac{10^4 \text{ V/cm}}{0.01 \text{ T}} = 10^6 \text{ m/sec}$$

To estimate the ∇B drift, we must find $v_\perp$ in the frame moving with the guiding center. Remember that in deriving $v_{\nabla B}$, $v_\perp$ was taken as the velocity in the undisturbed circular orbit. Here, the latter is moving with velocity v_E, so that it does not look circular in the lab frame. Nonetheless, it can still be decomposed into a circular motion with velocity $v_\perp$ plus an $E \times B$ drift of the guiding center. Consider the z component of velocity (along the wire). At point ① on the orbit, $v_z = v_E + v \cos\omega_c t = 0$, where $\cos\omega_c t = -1$, its maximum negative value; hence, $v_E = v_\perp$. The same result can be obtained by considering that at point ② $v_z = v_E + v_\perp (\cos\omega_c t = 1)$. The energy there, $\frac{1}{2}(mv_z^2)$, must equal the energy gained in falling a distance $2r_L$ in an electric field. Thus

$$\frac{1}{2}m(v_E + v_\perp)^2 2r_L eE = 2eE\frac{mv_\perp}{eB} = 2mv_\perp\frac{E}{B} = 2mv_\perp v_E$$

$$v_E^2 + 2v_\perp v_E + v_\perp^2 = 4v_\perp v_E \qquad (v_E - v_\perp)^2 = 0 \qquad v_E = v_\perp$$

Now we can calculate $v_{\nabla B}$:

$$v_{\nabla B} = \frac{1}{2}\frac{v_\perp^2}{\omega_c}\left|\frac{\nabla B}{B}\right| \qquad \omega_c = \frac{eB}{m} = \frac{(1.6\times 10^{-19})(10^{-2})}{(9.11\times 10^{-31})}$$

$$= 1.76\times 10^9\ \mathrm{sec}^{-1}$$

$$\frac{dB}{dr} = \frac{I(-1)10^{-4}}{r^2} = -\frac{B}{r} \qquad \left|\frac{\nabla B}{B}\right| = 10^2\ \mathrm{m}^{-1}$$

$$v_{\nabla B} = \frac{1}{2}\frac{v_E^2}{\omega_c} = \frac{1}{2}\frac{10^{16}}{1.8\times 10^9} = \underline{2.8\times 10^4\ \mathrm{m/sec}}$$

This amounts to a slowing down of the v_E drift due to a distortion of the orbit into a hairpin shape $\in$ because of the change in Larmor radius. The *undisturbed* orbit is the path taken by the valve on a bicycle wheel as it rolls along:

Finally, we note that the finite Larmor radius correction to v_E is negligible:

$$\frac{1}{4}r_L^2\nabla^2\frac{E}{B} \simeq \frac{1}{4}\frac{r_L^2}{r^2}\frac{E}{B}$$

$$r_L = \frac{(9.11\times 10^{-31})(10^6)}{(1.6\times 10^{-19})(0.01)} = 5.7\times 10^{-4}\ \mathrm{m}$$

$$r \simeq 10^{-2}\ \mathrm{m} \quad \therefore \quad \frac{1}{4}\frac{r_L^2}{r^2} = 0.08\%$$

2-12. Let all velocities refer to the midplane, and let subscripts i and f refer to initial and final states (before and after acceleration).

(a) Given: $R_m = 5$, $v_{\perp i} = v_{\parallel i}$ since μ is conserved, $v_{\perp f} = v_{\perp i}$, and only $v_\parallel$ will increase. It will increase until the pitch angle θ reaches the loss cone:

$$\sin^2\theta_m = \frac{v_{\perp f}^2}{v_{\perp f}^2 + v_{\parallel f}^2} = \frac{1}{1 + v_{\parallel f}^2/v_{\perp i}^2} = \frac{1}{R_m} = \frac{1}{5}$$

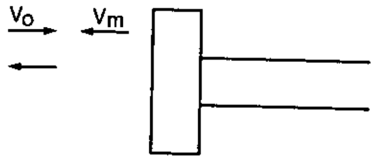
Hence $v_{\parallel f}^2/v_{\perp i}^2 = 4$, $v_{\parallel f} = 2v_{\perp i}$. Energy is

$$E_f = \tfrac{1}{2}M(v_{\parallel f}^2 + v_{\perp f}^2) = \tfrac{1}{2}M(4+1)v_{\perp i}^2 = \tfrac{5}{2}mv_{\perp i}^2$$

$$E_i = \tfrac{1}{2}M(v_{\parallel i}^2 + v_{\perp i}^2) = \tfrac{1}{2}M(1+1)v_{\perp i}^2 = Mv_{\perp i}^2$$

$$\therefore \quad E_f = 2.5E_i = (2.5)(1) = \underline{2.5\ \mathrm{keV}}$$

(b) (1) Let particle have $v_0 > 0$ and hit piston moving at velocity $v_m < 0$. *In the frame of the piston*, the particle bounces elastically and comes off with its initial velocity, but in the opposite direction. Let $'$ refer to the frame of the piston. Initial and final velocities in this frame are



$$v_i' = v_0 - v_m \qquad v_f' = -(v_0 - v_m)$$

(Note: v_m is negative.) Transforming back to lab frame,

$$v_f = v_f' + v_m = -v_0 + 2v_m$$

Since v_m is negative, the change in velocity is $2|v_m|$. QED

(2) At each bounce, the change in momentum is $\Delta p_{\parallel} = 2m|v_m|$. If N is the number of bounces, $p_{\parallel f} = p_{\parallel i} + N\Delta p$. Thus

$$N = \frac{p_{\parallel f} - p_{\parallel i}}{\Delta p} = \frac{v_{\parallel f} - v_{\parallel i}}{2v_m} = \frac{2v_{\perp i} - v_{\perp i}}{2v_m} = \frac{1}{2}\frac{v_{\perp i}}{vm}$$

$$E_i = Mv_{\perp i}^2 = 1 \text{ keV} = (10^3)(1.6 \times 10^{-19}) = 1.6 \times 10^{-16}\text{ J}$$

$$\therefore \; v_{\perp i} = \left(\frac{1.6 \times 10^{-16}}{1.67 \times 10^{-27}}\right)^{1/2} = 3.1 \times 10^5 \text{ m/sec}$$

$$v_m = 10^4 \text{ m/sec}$$

$$\therefore \; N = \frac{1}{2}\frac{(3 \times 10^5)}{10^4} = 15 \text{ bounces}$$

(3) Average $v_{\parallel}$ is

$$\bar{v} = \tfrac{1}{2}(v_{\parallel i} + v_{\parallel f}) = \tfrac{1}{2}(v_{\perp i} + 2v_{\perp i})$$
$$= \tfrac{3}{2}v_{\perp i} = 4.6 \times 10^5$$
$$L = 10^{13} \text{ m}$$
$$\therefore \; t = \frac{NL}{\bar{v}} = \frac{(15)(10^{13})}{4.6 \times 10^5} = 3.2 \times 10^8 \text{ sec}$$
$$(= 10 \text{ y})$$

However, L changes during this time by a distance

$$\Delta L = 2v_m t = (2)(10^4)(3.2 \times 10^8) = 6.4 \times 10^{12} \text{ m}$$

so that actual time is more like 2.5×10^8 sec. Since only factor-of-two accuracy is required, it is not necessary to sum the series—the above answer of 3.2×10^8 sec will do.

2-13. **(a)** $\int v_\parallel\, ds \simeq v_\parallel L = \text{constant} \quad \therefore \quad \dot{v}_\parallel L + v_\parallel \dot{L} = 0$

(b)

$$\frac{\dot{v}_\parallel}{v_\parallel} = -\frac{\dot{L}}{L} \qquad \dot{v}_\parallel \simeq \frac{\Delta v_\parallel}{T} = \frac{v_\parallel}{L}(-\dot{L})$$

$$T \simeq \frac{\Delta v_\parallel}{\bar{v}_\parallel}\frac{L}{-\dot{L}} = \frac{2v_{\perp i} - v_{\perp i}}{\frac{1}{2}(2v_{\perp i} + v_{\perp i})}\frac{L}{2v_m} = \frac{2}{3}\frac{10^{13}}{2 \times 10^4}$$
$$= \underline{3.3 \times 10^8 \text{ sec}}$$

2-14. As B increases, Maxwell's equation $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$ predicts an E-field. This induced E-field has a component along $\mathbf{v}$ and accelerates the particle. If B increases slowly and adiabatically, E will be small; but the integrated effect over many Larmor periods will be finite. The invariance of μ allows us to calculate the energy increases without doing this integration.

3-1. $\partial\sigma/\partial t + \nabla \cdot \mathbf{j} = 0$, where $\mathbf{j} = \mathbf{j}_P = (\rho/B^2)\dot{\mathbf{E}}$. Hence, $\dot{\sigma} = -\nabla \cdot [(\rho/B^2)\dot{\mathbf{E}}]$. The time derivative of Poisson's equation is $\nabla \cdot \dot{\mathbf{E}} = \dot{\sigma}/\epsilon_0$

$$\therefore \quad \nabla \cdot \dot{\mathbf{E}} = -\left(\frac{1}{\epsilon_0}\right)\nabla \cdot \left(\frac{\rho}{B^2}\right)\dot{\mathbf{E}} \qquad \nabla \cdot \left(1 + \frac{\rho}{\epsilon_0 B^2}\right)\dot{\mathbf{E}} = 0$$

Assuming the dielectric constant ϵ to be constant in time, we have $\nabla \cdot \dot{\mathbf{D}} = \nabla \cdot (\epsilon \dot{\mathbf{E}}) = 0$. By comparison, $\epsilon = 1 + \rho/\epsilon_0 B^2$.

3-2.

$$\epsilon \simeq 1 + \frac{nM}{\epsilon_0 B^2} \simeq \frac{\Omega_p^2}{\Omega_c^2} = \frac{ne^2}{\epsilon_0 M}\frac{M^2}{e^2 B^2} = \frac{nM}{\epsilon_0 B^2}$$

True if $\epsilon \gg 1$.

3-3. Take divergence of Eqs. [3-56] and [3-58]:

$$\nabla \cdot (\nabla \times \mathbf{E}) = -\nabla \cdot \dot{\mathbf{B}} = 0 \quad \therefore \quad \frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = 0$$

$\therefore \quad \nabla \cdot \mathbf{B} = 0$ if it is initially zero. This is Eq. [3-57],

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0 = \mu_0[q_i \nabla \cdot (n_i \mathbf{v}_i) + q_e \nabla \cdot (n_e \mathbf{v}_e)] + \frac{\nabla \cdot \dot{E}}{c^2}$$

from Eq. [3-60], $\nabla \cdot (n_i \mathbf{v}_i) = -\dot{n}_i$, $\nabla \cdot (n_e \mathbf{v}_e) = -\dot{n}_e$

$$\therefore \quad \mu_0(-q_i \dot{n}_i - q_e \dot{n}_e) + \frac{\nabla \cdot \dot{\mathbf{E}}}{c^2} = 0$$

$$\frac{\partial}{\partial t}\left[\nabla \cdot E - \frac{1}{\epsilon_0}(n_i q_i + n_e q_e)\right] = 0$$

If [ ] = 0 initially, $\nabla \cdot \mathbf{E} = (1/\epsilon_0)(n_i q_i + n_e q_e)$. This is Eq. [3-55].

3-4.

$$\mathbf{j}_D = (KT_i + KT_e)\frac{\mathbf{B}\times\nabla n}{B^2} \propto \frac{KT}{e}\frac{ne}{BL}$$

Since $KT \propto e\phi$ and $E \propto -\phi/L$, $KT/eL \propto E$ $\therefore$ $j_D \propto neE/B \propto nev$, since $E/B = v_E$.

3-5. Let j_D be constant in the box of width L. $\Delta n = n'L$, $|J_D| = |\Delta nev_y| = |n'Lev_y|$: from the difference between the currents on the two walls. This current J_D is over a box of width L, so the equivalent current density is

$$|j_D| = |J_D|/L = |n'ev_y|$$

Equation [3-69] gives $|j_D| \simeq |KT\nabla n/B| = |KTn'/B|$; hence, once v_y is chosen so the two formulas agree for one value of L, they agree for all L, since L cancels out.

3-6. (a)

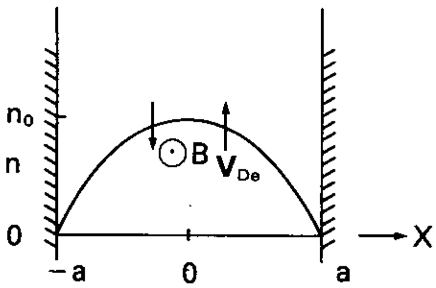
$$\mathbf{v}_{De} = -\frac{\gamma KT_e}{eB}\frac{\hat{z}\times\nabla n}{n}$$

Isothermal means $\gamma = 1$.

$$\nabla n = \hat{\mathbf{x}}\frac{\partial n}{\partial x} = -\frac{n_0 2x}{a^2}\hat{\mathbf{x}}$$

$$\mathbf{v}_{De} = \hat{\mathbf{y}}\frac{KT_e}{eB}\frac{2n_0}{a^2}\frac{x}{n_0}\left(1-\frac{x^2}{a^2}\right)^{-1} = \hat{\mathbf{y}}\frac{KT_e}{eB}\frac{2x}{a^2}\left(1-\frac{x^2}{a^2}\right)^{-1}$$

(b)



(c) $v_{De} = (2)/(0.2)\Lambda$

$$\Lambda^{-1} = \left|\frac{n'}{n}\right| = \frac{(2n_0/a^2)(a/2)}{n_0(1-a^2/4a^2)} = \frac{1/0.04}{3/4} = 33.3\ \mathrm{m}^{-1}$$

$$\therefore\ v_{De} = (10)(33.3) = 333\ \mathrm{m/sec}$$

3-7. $n = n_0 e^{-r^2/r_0^2} = n_0 e^{e\phi/KT_e}$

$$\phi = \frac{KT_e}{e}\ln\frac{n}{n_0} = \frac{KT_e}{e}\left(-\frac{r^2}{r_0^2}\right)$$

(a)

$$\mathbf{E} = -\frac{\partial\phi}{\partial r}\hat{\mathbf{r}} = \frac{KT_e}{e}\frac{2r}{r_0^2}\hat{\mathbf{r}}$$

$$\mathbf{v}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2} = -\frac{E_r}{B_z}\hat{\boldsymbol{\theta}} = -\hat{\boldsymbol{\theta}}\frac{KT_e}{eB}\frac{2r}{r_0^2}$$

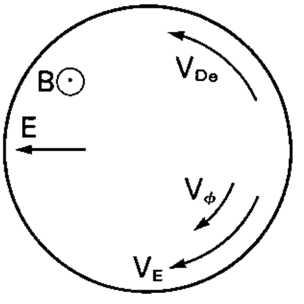
$$\mathbf{v}_{De} = -\frac{\mathbf{B}\times\nabla p}{enB^2} = -\frac{KT_e}{eB}\frac{\partial n/\partial r}{n}\hat{\boldsymbol{\theta}} = -\hat{\boldsymbol{\theta}}\frac{KT_e}{eB}\frac{\partial}{\partial r}(\ln n)$$

$$= -\hat{\boldsymbol{\theta}}\frac{KT_e}{eB}\frac{\partial}{\partial r}\left(\frac{-r^2}{r_0^2}\right) = \hat{\boldsymbol{\theta}}\frac{KT_e}{eB}\frac{2r}{r_0^2} = -\mathbf{v}_E \qquad \text{QED}$$

(b) From (a), the rotation frequency is constant whether we take $\mathbf{v}_E$, $\mathbf{v}_{De}$, $\mathbf{v}_{Di}$, or any combination thereof, since $\omega = v_\theta/r$ and $v_\theta \propto r$.

(c) In lab frame,

$$\mathbf{v} = \mathbf{v}_\phi + \mathbf{v}_E = 0.5\mathbf{v}_{De} + (-\mathbf{v}_{De})$$
$$= -\tfrac{1}{2}\mathbf{v}_{De}$$



3-8. (a)

$$j_D = ne(v_{Di} - v_{De}) = -\hat{\theta}\frac{n_0(KT_e + KT_i)}{B}\cdot\frac{2r}{r_0^2}e^{-r^2/r_0^2}$$

(b)

$$j_D = \frac{(10^{16})(0.5)(1.6\times10^{-19})}{0.4(r_0^2/2r)(2.718)} = 0.147\ \text{A/m}^2$$

or:

$$j_D = ne(|v_{De}| + |v_{Di}|)$$

$$|v_{De}| = |v_{Di}| = \frac{(KT)_{\text{eV}}}{B}\frac{2r}{r_0^2} = \frac{(0.25)2r}{0.4r_0^2} = 1.25\frac{r}{r_0^2}\ \text{m/sec}$$

Using $e = 1.6\times10^{-19}$ C, $\epsilon = 2.718$,

$$j_D = (10^{16})(1.6\times10^{-19})(2)(1.25)\frac{r\epsilon^{-1}}{r_0^2} = 0.147\ \frac{\text{A}}{\text{m}^2}$$

v_{De}, v_{Di}, or

(c) Since $\mathbf{v}_e = \mathbf{v}_E + \mathbf{v}_{De} = \mathbf{v}_E - \mathbf{v}_E = 0$ in the lab frame, the current is carried entirely by ions.

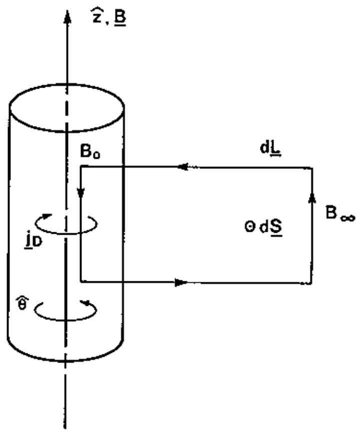
3-9.

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}_D$$

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \mu_0 \int \mathbf{j}_D \cdot d\mathbf{S}$$

$$\oint \mathbf{B} \cdot d\mathbf{L} = \mu_0 \int \mathbf{j}_D \cdot d\mathbf{S}$$

Choose a loop with one leg along the axis ($B = B_0$) and one leg far away, where $B = B_\infty$. Since $\mathbf{j}_D$ is in the $-\hat{\boldsymbol{\theta}}$ direction, we can choose the direction of integration $d\mathbf{L}$ as shown, so that $\mathbf{j}_D \cdot d\mathbf{S}$ is positive. There is no B_r $\therefore$



$$\oint \mathbf{B} \cdot d\mathbf{L} = (B_\infty - B_0)L$$

$$\mathbf{j}_D = -\hat{\boldsymbol{\theta}} \frac{n(KT_i + KT_e)}{B} \frac{2r}{r_0^2}$$

$$\int \mathbf{j}_D \cdot d\mathbf{S} = \frac{n_0(KT_i + KT_e)}{B_\infty r_0^2} \int_0^L \int_0^\infty e^{-r^2/r_0^2} 2r\, dr\, dz$$

$$= \frac{Ln_0(KT_i + KT_e)}{B_\infty} \left[-e^{-r^2/r_0} \right]_0^\infty = \frac{2Ln_0 KT}{B_\infty}$$

where $T_e = T_i$. In this integral, we have approximated $B(r)$ by B_∞, since B is not greatly changed by such a small j_D. Thus,

$$\Delta B = B_\infty - B_0 = \mu_0 \frac{2n_0 KT}{B_\infty}$$

$$= \frac{2(4\pi \times 10^{-7})(10^{16})(0.25)(1.6 \times 10^{-19})}{0.4}$$

$$= 2.5 \times 10^{-9}\,\mathrm{T}$$

4-1. (a) Solve for ϕ_1:

$$\phi_1 = \frac{KT_e}{e} \frac{n_1}{n_0} \frac{\omega + ia}{\omega^* + ia} \times \frac{\omega^* - ia}{\omega^* - ia}$$

$$= \frac{KT_e}{e} \frac{\omega\omega^* + a^2 + ia(\omega^* - \omega)}{\omega^{*2} + a^2} \frac{n_1}{n_0}$$

If n_1 is real,

$$\frac{Im(\phi_1)}{Re(\phi_1)} = \frac{a(\omega^* - \omega)}{\omega\omega^* + a^2} = \tan\delta$$

Hence,

$$\delta = \tan^{-1}\left[\frac{a(\omega^* - \omega)}{\omega\omega^* + a^2}\right]$$

(b) $n_1 = \bar{n}_1 e^{i(kx-\omega t)}$, while $\phi_1 = An_1 e^{i(kx-\omega t+\delta)}$, where A is a positive constant. For $\omega < \omega^*$, we have $\delta > 0$. Let the phase of n_1 be 0 at (x_0, t_0): $kx_0 - \omega t_0 = 0$. If ω and k are positive and x_0 is fixed, then the phase of ϕ_1 is 0 at $kx_0 - \omega t + \delta = 0$ or $t > t_0$. Hence ϕ_1 *lags* n_1 in time. If t_0 is fixed, $kx - \omega t_0 + \delta = 0$ at $x < x_0$, so ϕ_1 *lags* n_1 in space also (since $\omega/k > 0$ and the wave moves to the right, the leading wave is at larger x). If $k < 0$ and $\omega > 0$, the phase of ϕ_1 would be 0 at $x > x_0$; but since the wave now moves to the left, ϕ_1 still lags n_1.

4-2.

$$ikE_1 = \frac{1}{\epsilon_0}e(n_{i1} - n_{e1})$$

$$-i\omega m v_{e1} = -eE_1 \qquad \text{(electrons)}$$

$$-i\omega M v_{i1} = eE_1 \qquad \text{(ions)}$$

$$-i\omega n_{e1} = -ikn_0 v_{e1} \qquad \text{(electrons)}$$

$$-i\omega n_{i1} = -ikn_0 v_{i1} \qquad \text{(ions)}$$

$$n_{e1} = \frac{k}{\omega}n_0\left(\frac{-ie}{m\omega}\right)E_1 \qquad n_{i1} = \frac{k}{\omega}n_0\left(\frac{ie}{M\omega}\right)E_1$$

$$ikE_1 = \frac{1}{\epsilon_0}\frac{k}{\omega}n_0\frac{ie}{\omega}\left(\frac{1}{M} + \frac{1}{m}\right)E_1 = \frac{ikE_1}{\omega^2}(\Omega_p^2 + \omega_p^2)$$

$$\underline{\omega^2 = (\omega_p^2 + \Omega_p^2)}$$

4-3. Find ϕ_1, E_1, and v_1 in terms of n_1:

$$\text{Eq. [4-22]:}\quad v_1 = \frac{\omega}{k}\frac{n_1}{n_0}$$

$$\text{Eq. [4-23]:}\quad E_1 = \frac{ie}{\epsilon_0 k}n_1$$

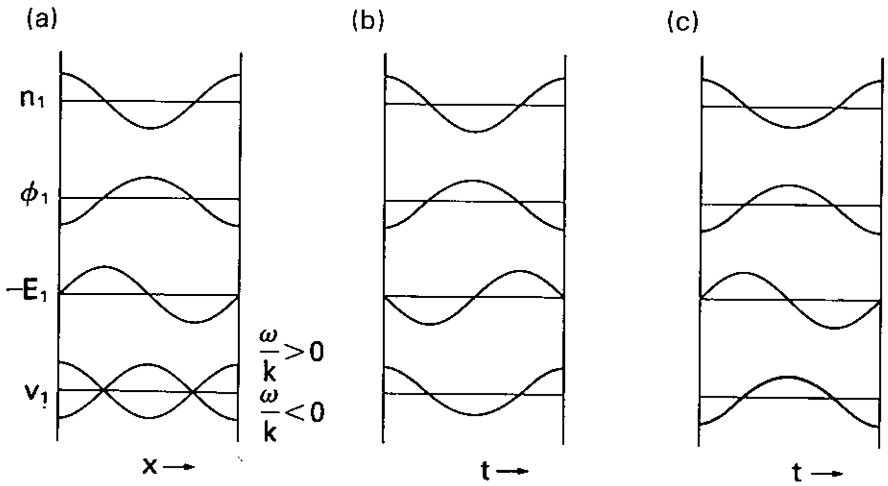
But $E_1 = -ik\phi_1$,

$$\therefore\quad \phi_1 = -\frac{e}{\epsilon_0 k^2}n_1$$

Hence, E_1 is 90° out of phase with n_1; ϕ_1 is 180° out of phase; and v_1 is either in phase or 180° out of phase, depending on the sign of ω/k. In (a), E_1 is found from the slope of the ϕ_1 curve, since $E_1 = -\partial\phi_1/\partial x$. In (b), $E_1/n_1 \propto i \times \text{sgn}(k)$ $\therefore\ \delta = \pm\pi/2$. If $\omega/k > 0$,

$$E_1 \propto \exp i(kx \pm |\omega|t \pm \pi/2)$$

the $\pm$ standing for the sign of k. Hence, E_1 leads n_1 by 90°. Opposite if $\omega/k < 0$.



4-4.

$$ikE_1 = -\frac{1}{\epsilon_0}en_1 = -\frac{1}{\epsilon_0}en_0\frac{k}{\omega}v_1 = -\frac{1}{\epsilon_0}en_0\frac{k}{\omega}\left(\frac{-ie}{m\omega}\right)E_1$$

$$ik\left(1 - \frac{n_0e^2}{\epsilon_0 m\omega^2}\right)E_1 = 0 \qquad \text{or} \qquad \nabla\cdot\left(1 - \frac{\omega_p^2}{\omega^2}\right)E_1 = 0$$

$$\therefore\ \epsilon = 1 - \frac{\omega_p^2}{\omega^2}$$

4-6. (a)

$$mn_0(-i\omega)v_1 = -en_0E_1 - mn_0\nu v_1$$

$$v_1\left(1 + \frac{i\nu}{\omega}\right) = \frac{ieE_1}{m\omega}$$

$$ikE_1 = -\frac{1}{\epsilon_0}en_1 \qquad n_1 = \frac{k}{\omega}n_0v_1 \qquad \text{(continuity)}$$

$$ikE_1 = -\frac{1}{\epsilon_0}e\frac{k}{\omega}n_0\frac{ieE_1}{m\omega}\left(1 + \frac{i\nu}{\omega}\right)^{-1}$$

$$\omega^2\left(1 + \frac{i\nu}{\omega}\right) = \omega_p^2 \qquad \underline{\omega^2 + i\nu\omega = \omega_p^2}$$

(b) Let $\omega = x + iy$. Then the dispersion relation is $x^2 - y^2 + 2ixy + i\nu x - \nu y = \omega_p^2$. We need the imaginary part: $2xy + \nu x = 0$, $y = (-1/2)\nu$ $\therefore$ $\mathrm{Im}(\omega) = -\nu/2$. Since $x = \mathrm{Re}(\omega)$, $\nu > 0$, and

$$E_1 \propto e^{-i\omega t} = e^{-i\omega t} e^{yt} = e^{-ixt} e^{-(1/2)\nu t}$$

the oscillation is damped in time.

4-7. $mn_0(-i\omega)\mathbf{v}_1 = en_0\mathbf{E}_1 - en_0(\mathbf{v}_1 \times \mathbf{B}_0)$. Take $\mathbf{B}_0$ in the $\hat{\mathbf{z}}$ direction and $\mathbf{E}_1$ and $\mathbf{k}$ in the $\hat{\mathbf{x}}$ direction. Then the y-component is

$$-i\omega m v_y = e v_x B_0 \qquad \frac{v_x}{v_y} = -i\frac{\omega}{\omega_c}$$

Since $\omega = \omega_h > \omega_c$, $|v_x/v_y| > 1$; and the orbit is elongated in the $\hat{\mathbf{x}}$ direction, which is the direction of $\mathbf{k}$.

4-8. (a)

$$\nabla \cdot \mathbf{E}_1 = -\frac{1}{\epsilon_0} e n_1 \qquad \mathbf{k} = k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}} \qquad E_y = k_y = 0$$

$$i(k_xE_x + k_zE_z) = -\frac{1}{\epsilon_0} e n_1$$

We need n_1:

$$\frac{\partial n_1}{\partial t} + n_0\nabla \cdot \mathbf{v}_1 = 0 \qquad -i\omega n_1 + n_0 i(k_xv_x + k_zv_z) = 0$$

We need v_x, v_z:

$$Mn_0(-i\omega)\mathbf{v}_1 = -en_0\mathbf{E}_1 - en_0(\mathbf{v}_1 \times \mathbf{B}_0)$$

$$x\text{-component:} \quad v_x = -\frac{ie}{m\omega}E_x - \frac{i\omega_c}{\omega}v_y$$

$$y\text{-component:} \quad v_y = 0 + \frac{i\omega_c}{\omega}v_x$$

$$v_x = -\frac{ie}{m\omega}E_x + \frac{\omega_c^2}{\omega^2}v_x = \frac{-ie}{m\omega}E_x\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}$$

$$z\text{-component:} \quad v_z = -\frac{ie}{m\omega}E_z$$

$$\text{Continuity:} \quad n_1 = \frac{n_0}{\omega}\left(\frac{-ie}{m\omega}\right)\left[k_xE_x\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1} + k_zE_z\right]$$

$$k_xE_x + k_zE_z = i\frac{en_0}{\epsilon_0\omega}\left(\frac{-ie}{m\omega}\right)\left[k_xE_x\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1} + k_zE_z\right]$$

$$k_x = k\sin\theta \qquad k_z = k\cos\theta$$

$$\therefore \quad E_1\sin^2\theta + kE_1\cos^2\theta = \frac{\omega_p^2}{\omega^2}\left[kE_1\sin^2\theta\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1} + kE_1\cos^2\theta\right]$$

$$1 = \frac{\omega_p^2}{\omega^2}\left[\sin^2\theta\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1} + \cos^2\theta\right]$$

$$1 - \frac{\omega_c^2}{\omega^2} = \frac{\omega_p^2}{\omega^2}\left[1 - \cos^2\theta + \left(1 - \frac{\omega_c^2}{\omega^2}\right)\cos^2\theta\right]$$

$$\omega^2 - \omega_c^2 - \omega_p^2 = -\frac{\omega_p^2\omega_c^2}{\omega^2}\cos^2\theta$$

$$\omega^2(\omega^2 - \omega_h^2) + \omega_p^2\omega_c^2\cos^2\theta = 0 \qquad \text{QED}$$

(b)

$$\omega^4 - \omega_h^2\omega^2 + \omega_p^2\omega_c^2\cos^2\theta = 0$$

$$2\omega^2 = \omega_h^2 \pm (\omega_h^4 - 4\omega_p^2\omega_c^2\cos^2\theta)^{1/2}$$

For $\theta \to 0$, $\cos^2\theta \to 1$,

$$2\omega^2 = \omega_h^2 \pm [(\omega_p^2 + \omega_c^2)^2 - 4\omega_p^2\omega_c^2]^{1/2}$$

$$= \omega_p^2 + \omega_c^2 \pm (\omega_p^2 - \omega_c^2)$$

$$\omega^2 = \omega_p^2, \omega_c^2$$

The $\omega = \omega_p$ root is the usual Langmuir oscillation. The $\omega = \omega_c$ root is spurious because at $\theta \to 0$, B_0 does not enter the problem. For $\theta \to \pi/2$, $\cos^2\theta \to 0$, $2\omega^2 = \omega_h^2 \pm \omega_h^2$, $\omega = 0, \omega_h$. The $\omega = \omega_h$ root is the usual upper hybrid oscillation. The $\omega = 0$ root has no physical meaning, since on oscillating perturbation was assumed.

(c)

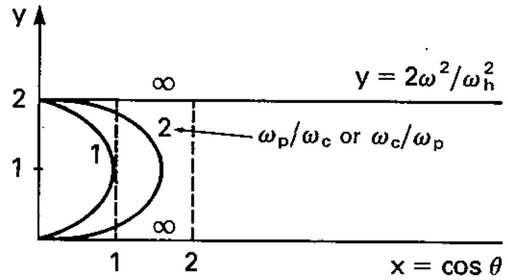
$$\omega^4 - \omega_h^2\omega^2 + \tfrac{1}{4}\omega_h^4 = \tfrac{1}{4}\omega_h^4 - \omega_p^2\omega_c^2\cos^2\theta$$

$$(\omega^2 - \tfrac{1}{2}\omega_h^2) + (\omega_p\omega_c\cos\theta)^2 = (\tfrac{1}{2}\omega_h)^2$$

$$(y - 1)^2 + \frac{x^2}{a^2} = 1 \qquad \text{QED}$$

(d)

ω_p/ω_c	$a = \frac{1}{2}(\omega_c/\omega_p + \omega_p/\omega_c)$
1	1
2	5/4
∞	∞



(e)

$$\omega^2 = \tfrac{1}{2}(\omega_p^2 + \omega_c^2) \pm [(\omega_p^2 + \omega_c^2)^2 - 4\omega_p^2\omega_c^2 \cos^2\theta]^{1/2}$$

Lower root: Take (−) sign; ω is maximum when $\cos^2\theta$ is maximum (=1). Thus

$$\begin{aligned} \omega_-^2 &< \tfrac{1}{2}[(\omega_p^2 + \omega_c^2) - |\omega_p^2 - \omega_c^2|] \\ &= \omega_c^2 \qquad \text{if } \omega_p > \omega_c \\ &= \omega_p^2 \qquad \text{if } \omega_c > \omega_p \end{aligned}$$

Upper root: Take (+) sign; ω is maximum when $\cos^2\theta = 0$, $\omega^2 = \omega_h^2$. Thus $\omega_+^2 < \omega_h^2$. This root is minimum when $\cos^2\theta = 1$; thus

$$\begin{aligned} \omega_+^2 &> \tfrac{1}{2}[(\omega_p^2 + \omega_c^2) + |\omega_p^2 - \omega_c^2|] \\ &= \omega_p^2 \qquad \text{if } \omega_p > \omega_c \\ &= \omega_c^2 \qquad \text{if } \omega_c > \omega_p \end{aligned}$$

4-10. Use V_+, N_+ for proton velocity and density
V_-, N_- for antiprotons
v_-, n_- for electrons
v_+, n_+ for positrons

(a)

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{\dot{\mathbf{E}}}{c^2} \qquad \nabla \times \nabla \times \mathbf{E} = -\left(\mu_0 \dot{\mathbf{j}} + \frac{\ddot{\mathbf{E}}}{c^2}\right)$$

$$-(\mathbf{k} \times \mathbf{k} \times \mathbf{E}) = -\left[\mu_0 n_0 e(\dot{\mathbf{v}}_+ - \dot{\mathbf{v}}_-) - \frac{\omega^2}{c^2}\mathbf{E}\right]$$

$$= k^2\mathbf{E} - \mathbf{k}(\overset{0}{\cancel{\mathbf{k}\cdot\mathbf{E}}})$$

$$(\omega^2 - c^2k^2)\mathbf{E} = \frac{1}{\epsilon_0} n_0 e(\dot{\mathbf{v}}_+ - \dot{\mathbf{v}}_-)$$

$$mn_0\mathbf{v}_\pm = \pm en_0\mathbf{E} \qquad \dot{v}_\pm = \pm\frac{e}{m}\mathbf{E}$$

$$\omega^2 - c^2k^2 = \frac{1}{\epsilon_0} n_0 e \frac{e}{m}(1 + 1) = 2\omega_p^2$$

$$\omega_p^2 = \frac{n_0 e^2}{\epsilon_0 m} \qquad \underline{\omega^2 = 2\omega_p^2 + c^2k^2}$$

(Or the 2 can be incorporated into the definition of ω_p.)

(b) $\nabla \cdot \mathbf{E}_1 = (1/\epsilon_0)(N_+ - N_- + n_+ - n_-)_1$, where $n_+ = n_0\, e^{-e\phi/KT_+}$, $n_- = n_0\, e^{e\phi/KT_-}$. Let $T_+ = T_- = T_e$ $n_{1\pm} = \mp\, n_0 e\phi/KT_e$. Note: $N_{0\pm} = n_{0\pm} \equiv n_0$.

$$\frac{\partial N_\pm}{\partial t} + N_{0\pm}\nabla \cdot V_\pm = 0 \qquad N_{1\pm} = N_{0\pm}\frac{k}{\omega}V_\pm = n_0\frac{k}{\omega}V_\pm$$

$$M(-i\omega)V_\pm = \pm e\mathbf{E}_1 = \pm i\mathbf{k}e\phi \qquad (M_+ = M_- = M)$$

$$V_\pm = \pm\frac{k}{\omega}\frac{e\phi}{M} \qquad N_{1\pm} = \pm\frac{k^2}{\omega^2}\frac{n_0 e\phi}{M}$$

$$\nabla \cdot \mathbf{E}_1 = k^2\phi = \frac{e}{\epsilon_0}\left(\frac{k^2}{\omega^2} + \frac{k^2}{\omega^2}\right)\frac{n_0 e\phi}{M} + \frac{e}{\epsilon_0}(-n_0 - n_0)\frac{e\phi}{KT_e}$$

$$= \frac{n_0 e^2}{\epsilon_0 M}\frac{2k^2}{\omega^2}\phi - \frac{n_0 e^2}{\epsilon_0 kT_e}2\phi = 2\phi\left(\Omega_p^2\frac{k^2}{\omega^2} - \frac{1}{\lambda_D^2}\right)$$

$$k^2\lambda_D^2 + 2 = \frac{2k^2}{\omega^2}\Omega_p^2\lambda_D^2 = \frac{2k^2}{\omega^2}v_s^2 \qquad v_s^2 \equiv \frac{kT_e}{M}$$

$$\frac{\omega^2}{k^2} = \frac{2v_s^2}{2 + k^2\lambda_D^2} = \frac{v_s^2}{1 + (1/2)k^2\lambda_D^2} \qquad \lambda_D \equiv \left(\frac{kT_e\epsilon_0}{n_0 e^2}\right)^{1/2}$$

4-11.

$$\tilde{n} = \frac{ck}{\omega} \qquad \omega^2 = \omega_p^2 + c^2k^2 \qquad \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} = \epsilon$$

$$\therefore\ \tilde{n} = \sqrt{\epsilon}$$

4-12. In $\nabla \times \mathbf{B} = \mu_0\mathbf{j}_1$, $\mathbf{j}_1$ is the current carried by electrons only, since Cl^- ions are too heavy to move appreciably in response to a signal at microwave frequencies. Hence,

$$j_1 = -n_{0e}ev_e = -(1-\kappa)n_0 ev_{e1}$$

If ω_p is defined with n_0 (i.e., $\omega_p^2 = n_0e^2/\epsilon_0 m$), the dispersion relation becomes

$$\frac{c^2k^2}{\omega^2} = 1 - (1-\kappa)\frac{\omega_p^2}{\omega^2}$$

Cutoff occurs for $f = (1-\kappa)^{1/2}f_p = (0.4)^{1/2}(9)(n_0)^{1/2}$, where

$$f = \frac{c}{\lambda} = \frac{3 \times 10^{10}}{3} = 10^{10}$$

Thus

$$n_0 = \left[\frac{10^{10}}{(0.63)(9)}\right]^2 = 3.1 \times 10^{18}\ \text{m}^{-3}$$

4-13. **(a)** Method 1: Let N = No. of wavelengths in length $L = 0.08$ m, N_0 = No. of wavelengths in absence of plasma.

$$N = \frac{L}{\lambda} \qquad N_0 = \frac{L}{\lambda_0} \qquad \lambda = \frac{2\pi}{k} \qquad \frac{ck}{\omega} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}$$

$$\Delta N = N_0 - N = \frac{L}{\lambda_0} - \frac{Lk}{2\pi} = \frac{L}{\lambda_0} - \frac{L}{2\pi}\frac{\omega}{c}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}$$

$$\frac{\omega}{2\pi c} = \frac{1}{\lambda_0} \quad \therefore \quad \Delta N = \frac{L}{\lambda_0}\left[1 - \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}\right] = 0.1$$

$$\frac{L}{\lambda_0} = \frac{0.08}{0.008} = 10$$

$$\therefore \quad \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} = 1 - 10^{-2} \qquad 1 - \frac{f_p^2}{f^2} = 1 - (2 \times 10^{-2})$$

$$f_p^2 = f^2 \times 2 \times 10^{-2} = \left(\frac{c}{\lambda_0}\right)^2 2 \times 10^{-2} = 2.8 \times 10^{19}$$

$$n = \frac{2.8 \times 10^{19}}{(9)^2} = \underline{3.5 \times 10^{17}\ \mathrm{m}^{-3}}$$

Method 2: Let k_0 = free-space k. The phase shift is

$$\Delta\phi = \int_0^L \Delta k\, dx = (k_0 - k)L = (0.1)2\pi$$

This leads to the same answer.

(b) From above, ΔN is small if ω_p^2/ω^2 is small; hence expand square root:

$$\Delta N \approx \frac{L}{\lambda_0}\left[1 - \left(1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right)\right] = \frac{L}{\lambda_0}\frac{1}{2}\frac{\omega_p^2}{\omega^2} \propto n \qquad \text{QED}$$

4-14. From Eq. [4-101a], we have for the X-wave

$$(\omega^2 - \omega_h^2)E_x + i\frac{\omega_p^2\omega_c}{\omega}E_y = 0$$

At resonance, $\omega = \omega_h$ $\therefore$ $E_y = 0$, $\mathbf{E} = E_x\hat{\mathbf{x}}$. Since $\mathbf{k} = k_x\hat{\mathbf{x}}$, $\mathbf{E} \parallel \mathbf{k}$, and the wave is longitudinal and electrostatic.

4-15. Since $\omega_h^2 = \omega_c^2 + \omega_p^2$, clearly $\omega_p < \omega_h$. Further,

$$\begin{aligned}\omega_L &= \tfrac{1}{2}[-\omega_c + (\omega_c^2 + 4\omega_p^2)^{1/2}] \\ &< \tfrac{1}{2}[-\omega_c + (\omega_c^2 + 4\omega_c\omega_p + 4\omega_p^2)^{1/2}] \\ &= \tfrac{1}{2}[-\omega_c + (\omega_c + 2\omega_p)] = \omega_p \quad \therefore \quad \omega_L < \omega_p\end{aligned}$$

Also,

$$\omega_R = \tfrac{1}{2}[\omega_c + (\omega_c^2 + 4\omega_p^2)^{1/2}] > \omega_c$$

and

$$\omega_R^2 - \omega_R\omega_c - \omega_p^2 = 0 \quad \text{(Eq. [4-107])}$$

$$\therefore \; \omega_R^2 = \omega_R\omega_c + \omega_p^2 > \omega_c^2 + \omega_p^2 = \omega_h^2$$

4-17. **(a)** Multiply Eq. [4-112b] by i and add to Eq. [4-112a]:

$$(\omega^2 - c^2k^2 - \alpha)(E_x + iE_y) + \alpha\frac{\omega_c}{\omega}(E_x + iE_y) = 0$$

Now subtract from Eq. [4-112a]:

$$(\omega^2 - c^2k^2 - \alpha)(E_x - iE_y) - \alpha\frac{\omega_c}{\omega}(E_x - iE_y) = 0$$

Thus,

$$F(\omega) = \omega^2 - c^2k^2 - \alpha(1 + \omega_c/\omega)$$

$$G(\omega) = \omega^2 - c^2k^2 - \alpha(1 - \omega_c/\omega)$$

Since

$$\alpha \equiv \frac{\omega_p^2}{(1 - \omega_c^2/\omega^2)}$$

$$F(\omega) = \omega^2\left(1 - \frac{\omega_p^2/\omega^2}{1 - \omega_c/\omega} - \frac{c^2k^2}{\omega^2}\right)$$

$$G(\omega) = \omega^2\left(1 - \frac{\omega_p^2/\omega^2}{1 + \omega_c/\omega} - \frac{c^2k^2}{\omega^2}\right)$$

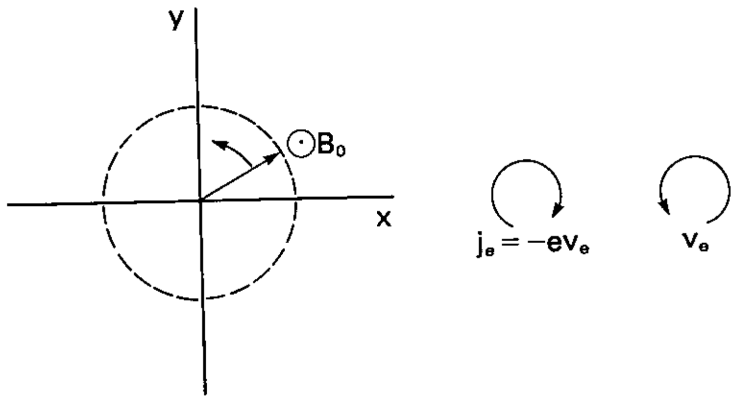
From Eqs. [4-116] and [4-117],

$$F(\omega) = 0 \text{ for the } R \text{ wave} \quad \text{and}$$

$$G(\omega) = 0 \text{ for the } L \text{ wave}$$

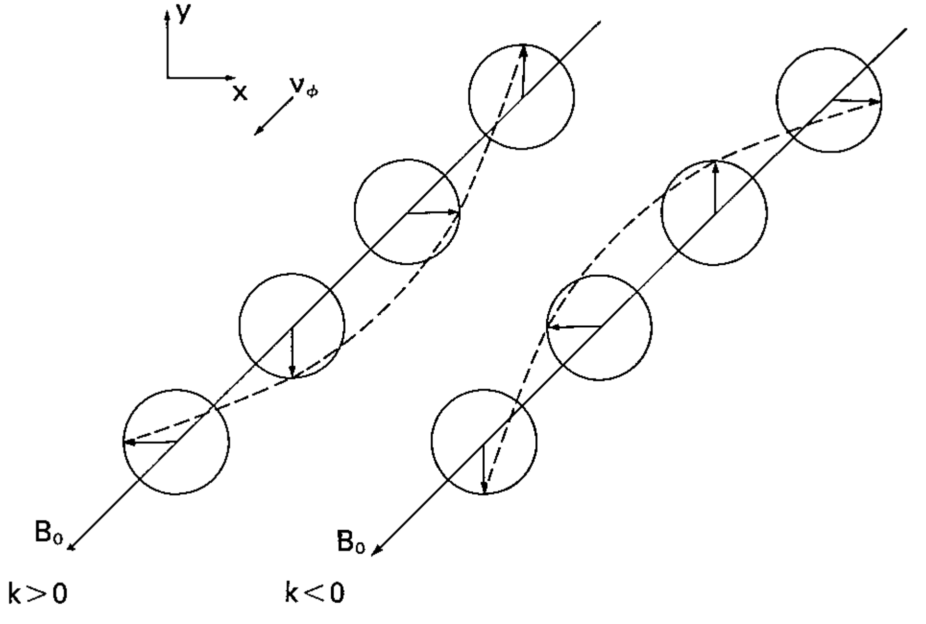
(b) $E_x = -iE_y \;\therefore\; E_y = iE_x$. Let $E_x = f(z)\,e^{-i\omega t}$. Then

$$E_y = f(z)i\,e^{-i\omega t} = f(z)\,e^{-i\omega t + i(\pi/2)} = f(z)\,e^{-i[\omega t - (\pi/2)]}$$



E_y *lags* E_x by 90°. Hence E rotates counterclockwise on this diagram. This is the same way electrons gyrate in order to create a clockwise current and generate a B-field opposite to $\mathbf{B}_0$. For the L wave, $E_y = -iE_x$ so that $E_y = f(z)\,e^{-i(\omega t+\pi/2)}$ and E_y *leads* E_x by 90°.

(c) For an R-wave, $E_y = iE_x$. The space dependence is $E_x = f(t)\,e^{ikz}$, $E_y = f(t)i\,e^{ikz} = f(t)\,e^{i(kz+\pi/2)}$. For $k > 0$, E_y *leads* E_x (has the same phase at smaller z). For $k < 0$, E_y *lags* E_x (has the same phase at larger z).



4-19.

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2/\omega^2}{1-\omega_c/\omega} \qquad c^2 v_\phi^{-2} = 1 - \frac{\omega_p^2/\omega^2}{1-\omega_c/\omega}$$

$$c^2(-2)v_\phi^{-3}\frac{dv_\phi}{d\omega} = -\omega_p^2\frac{-1}{(\omega^2-\omega\omega_c)^2}(2\omega-\omega_c) = 0$$

$$\therefore\ 2\omega - \omega_c = 0 \qquad \omega = \tfrac{1}{2}\omega_c$$

At $\omega = \frac{1}{2}\omega_c$,

$$\frac{c^2}{v_\phi^2} = 1 - \frac{\omega_p^2}{\frac{1}{4}\omega_c^2 - \frac{1}{2}\omega_c^2} = 1 + \frac{4\omega_p^2}{\omega_c^2} > 1$$

$$\therefore\ v_\phi < c.$$

4-20.

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2/\omega^2}{1 - \omega_c/\omega} \qquad c^2k^2 = \omega^2 - \frac{\omega\omega_p^2}{\omega - \omega_c}$$

$$c^2 2k\,dk = 2\omega\,d\omega - \frac{(\omega - \omega_c) - \omega}{(\omega - \omega_c)^2}\,\omega_p^2\,d\omega$$

$$= \left[2\omega + \frac{\omega_c\omega_p^2}{(\omega - \omega_c)^2}\right] d\omega$$

$$\frac{d\omega}{dk} = \frac{kc^2}{\omega + \omega_c\omega_p^2/2(\omega - \omega_c)^2} \approx \frac{kc^2}{\omega + \omega_p^2/2\omega_c} \qquad \text{if } \omega \ll \omega_c$$

But

$$ck = \left(\omega^2 - \frac{\omega_p^2}{1 - \omega_c/\omega}\right)^{1/2} \approx \left(\omega^2 + \frac{\omega\omega_p^2}{\omega_c}\right)^{1/2} \qquad \text{if } \omega \ll \omega_c$$

$$\therefore\ \frac{d\omega}{dk} = c\,\frac{(\omega^2 + \omega\omega_p^2/\omega_c)^{1/2}}{\omega + \omega_p^2/2\omega_c} = c\,\frac{(1 + \omega_p^2/\omega\omega_c)^{1/2}}{1 + \omega_p^2/2\omega\omega_c}$$

To prove the required result, one must also assume $v_\phi^2 \ll c^2$, as is true for whistlers, so that $\omega_p^2/\omega\omega_c \ll 1$ (from line 1). Hence

$$\frac{d\omega}{dk} \approx 2c\left(\frac{\omega\omega_c}{\omega_p^2}\right)^{1/2} \propto \omega^{1/2}$$

4-21.

$$(\omega^2 - c^2k^2)\mathbf{E}_1 = \frac{1}{\epsilon_0}\,i\omega\,\mathbf{j}_1 \qquad \text{(Eq. [4-81])}$$

$$\mathbf{j}_1 = n_0 e(\mathbf{v}_p - \mathbf{v}_e) \qquad (v_p \text{ is the positron velocity})$$

From the equation of motion,

$$v_x = \frac{\pm ie}{m\omega}\left(E_x \pm \frac{i\omega_c}{\omega}E_y\right)\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}$$

$$v_y = \frac{\pm ie}{m\omega}\left(E_y \mp \frac{i\omega_c}{\omega}E_x\right)\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}$$

$$\therefore\ (\omega^2 - c^2k^2)E_x = \left(-\frac{1}{\epsilon_0}i\omega\right)(n_0 e)\left(\frac{ie}{m\omega}\right)(1 + 1)E_x\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}$$

$$= \frac{2\omega_p^2}{1 - \omega_c^2/\omega^2}E_x$$

the E_y term canceling out. Similarly,

$$(\omega^2 - c^2k^2)E_y = \frac{2\omega_p^2}{1 - \omega_c^2/\omega^2}E_y$$

the E_x term cancelling out. Both equations give

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{2\omega_p^2}{\omega^2 - \omega_c^2}$$

The R and L waves are degenerate and have the same phase velocities—hence, no Faraday rotation.

4-22. Since the phase difference between the R and L waves is twice the angle of rotation,

$$\int_0^L (k_L - k_R)\,dz = \pi$$

$$k_{R,L} = k_0\left(1 - \frac{\omega_p^2/\omega^2}{1 \pm \omega_c/\omega}\right)^{1/2}$$

To get a simple expression for $k_L - k_R$, we wish to expand the square root. Let us assume we can, and then check later for consistency:

$$k_{R,L} \approx k_0\left(1 - \frac{1}{2}\,\frac{\omega_p^2/\omega^2}{1 \pm \omega_c/\omega}\right)$$

$$k_L - k_R = \frac{1}{2}k_0\frac{\omega_p^2}{\omega^2}\left(\frac{1}{1-\omega_c/\omega} - \frac{1}{1+\omega_c/\omega}\right)$$

$$= \frac{1}{2}k_0\frac{\omega_p^2}{\omega^2}\,\frac{2\omega_c/\omega}{1-\omega_c^2/\omega^2}$$

$$\pi = L(k_L - k_R) = k_0L\frac{\omega_p^2\omega_c}{\omega}\,\frac{1}{\omega^2-\omega_c^2} \qquad k_0 = \frac{\omega}{c}$$

$$\omega_p^2 = \frac{\pi c}{L\omega_c}(\omega^2 - \omega_c^2) \qquad f_p^2 = \frac{c}{2L}\,\frac{f^2 - f_c^2}{f_c}$$

$$f_c = 2.8 \times 10^{10}(0.1)\ \text{Hz}$$

$$f = \frac{c}{\lambda_0} = \frac{3\times 10^8}{8\times 10^{-3}} = 3.75\times 10^{10}\ \text{Hz}$$

$$f_p^2 = \frac{(3\times 10^8)}{(2)(1)}\,\frac{(1.41\times 10^{21} - 7.8\times 10^{18})}{2.8\times 10^9}$$

$$= 7.5\times 10^{19} = 9^2 n$$

$$n = 9.3\times 10^{17}\ \text{m}^{-3}$$

To justify expansion, note that $f_c \ll f$, so that

$$\frac{\omega_p^2/\omega^2}{1\pm\omega_c/\omega} \approx \frac{f_p^2}{f^2} = \frac{7.5\times 10^9}{(3.75\times 10^{10})^2} = 0.05 \ll 1$$

4-24. 12.7°.

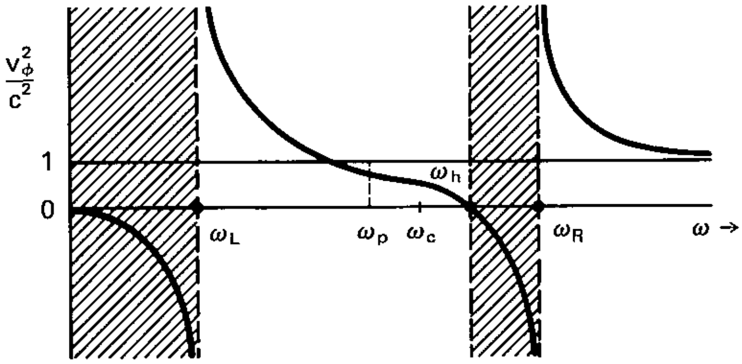
4-25. (a) The X-wave cutoff frequencies are given by Eq. (4-107). Thus,

$$\omega_p^2 = \omega(\omega \pm \omega_c) = \frac{4\pi n e^2}{m}$$

$$n_{cx} = \frac{m\omega}{4\pi e^2}(\omega + \omega_c)$$

We choose the (+) sign, corresponding to the L cutoff, because that gives the higher density.

(b)



The left branch is the one that has a cutoff at $\omega = \omega_L$. One might worry that this branch is inaccessible if the wave is sent in from outside the plasma. However, if ω is kept less than ω_c, the stopband between ω_h and ω_R is avoided completely.

4-28. (a)

$$f_p = 9\sqrt{n} = (9)(10^{15})^{1/2} = 2.85 \times 10^8 \text{ Hz}$$

$$f_c = 28 \text{ GHz/T} = (2.8 \times 10^{10}) \times (10^{-2}) = 2.8 \times 10^8 \text{ Hz}$$

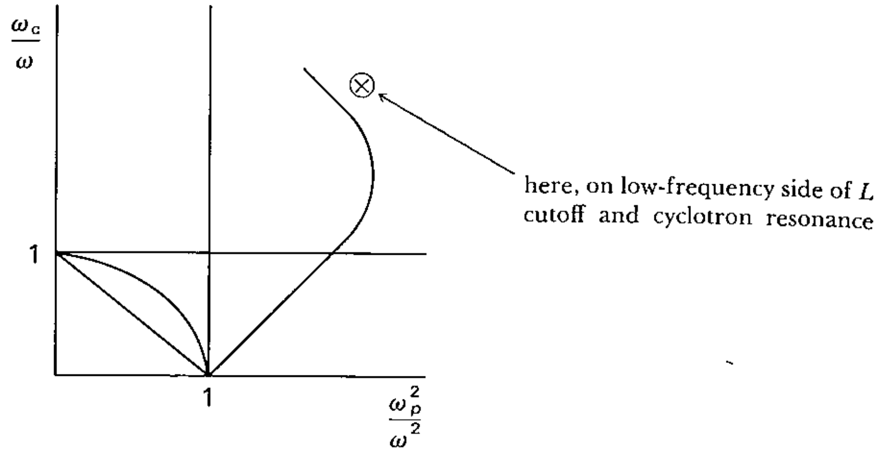
$$f = 1.6 \times 10^8 \text{ Hz} \;\therefore\; \omega_p/\omega > 1 \qquad \omega_c/\omega > 1$$

$$\omega_L = \tfrac{1}{2}[-\omega_c \pm (\omega_c^2 + 4\omega_p^2)^{1/2}] \approx \tfrac{1}{2}(-\omega_c + \sqrt{5}\omega_c)$$

$$= 0.62\omega_c \qquad \text{for } \omega_c \approx \omega_p$$

$$f_L = (0.62)(2.8 \times 10^8) = 1.73 \times 10^8 > f$$

Also, $f >$ all ion frequencies.



(b) The R-wave (whistler mode) is the only wave that propagates here.

4-29. (a)

$$v_A = \frac{B}{(\mu_0 n M)} = \frac{1}{[(1.26 \times 10^{-6})(10^{19})(1.67 \times 10^{-27})]^{1/2}}$$
$$= 6.9 \times 10^6 \text{ m/sec}$$

$$\Omega_c = \frac{eB}{M} = \frac{(1.6 \times 10^{-19})(1)}{(1.67 \times 10^{-27})} = 9.58 \times 10^7 \text{ rad/sec}$$

$$\omega = 0.1\Omega_c = 9.58 \times 10^6 \text{ rad/sec}$$

$$\omega = kv_A = 2\pi v_A/\lambda$$

If $\lambda = 2L$,

$$L = \frac{\pi v_A}{\omega} = \frac{\pi(6.9 \times 10^6)}{9.58 \times 10^6} = 2.26 \text{ m}$$

(b)

$$L \propto v_A/\omega \propto v_A/\Omega_c \propto B(nM)^{-1/2}B^{-1}M \propto (M/n)^{1/2}$$

$$\therefore\ L = (2.26)\left(\frac{133}{1}\right)^{1/2}\left(\frac{10^{19}}{10^{18}}\right)^{1/2} = 82 \text{ m}$$

This is why Alfvén waves cannot be studied in Q-machines, regardless of B.

4-30.

(a)

$$\omega^2 = \omega_p^2 + c^2k^2 \qquad 2\omega\, d\omega = c^2 2k\, dk$$

$$v_g = d\omega/dk = c^2k/\omega$$

$$\frac{ck}{\omega} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}$$

$$\therefore v_g = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} \approx c\left(1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right) \qquad \text{for } \omega^2 \gg \omega_p^2$$

$$v_g t = x \quad \therefore\ t = x/v_g$$

$$\frac{dt}{d\omega} = \frac{x}{c}\left(1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right)^{-2}\left(-\frac{\omega_p^2}{\omega^2}\right) \approx -\frac{x}{c}\frac{\omega_p^2}{\omega^3}$$

$$\therefore\ \frac{df}{dt} \approx -\frac{c}{x}\frac{f^3}{f_p^2}$$

(b)

$$x = \frac{cf^3}{f_p^2}\left(-\frac{df}{dt}\right)^{-1} = \frac{(3\times 10^8)(8\times 10^7)^3}{(9)^2(2\times 10^5)(5\times 10^6)} = 1.9\times 10^{18}\ \text{m}$$

$$= (1.9\times 10^{18})(3\times 10^{16})^{-1} = 63\ \text{parsec}$$

4-31. **(a)** Let $n_0^{(1)} = (1-\epsilon)n_0, \quad n_0^{(2)} = \epsilon n_0, \quad n_e = n_0\, e\phi/kT_e$

Poisson: $ikE_1 = k^2\phi = \dfrac{1}{\epsilon_0} e\,(n_i^{(1)} + n_i^{(2)} - n_e)$

(Assume $z_{1,2} = 1$, since the ion charge is not explicitly specified.)

Continuity: $n_1^{(1)} = (1-\epsilon)n_0 \dfrac{k}{\omega} v_1^{(1)}, \qquad n_1^{(2)} = \epsilon n_0 \dfrac{k}{\omega} v_1^{(2)}$

Equation of motion: $v_1^{(j)} = \dfrac{e}{M_j}\dfrac{k}{\omega}\phi\left(1 - \dfrac{\Omega_{cj}^2}{\omega^2}\right)^{-1} \qquad$ (Eq. [4-68])

$$\therefore\ k^2\phi = \frac{e}{\epsilon_0}\left[(1-\epsilon)n_0\frac{k^2}{\omega^2}\frac{e}{M_1}\left(1 - \frac{\Omega_{c1}^2}{\omega^2}\right)^{-1}\right.$$

$$\left. + \epsilon n_0 \frac{k^2}{\omega^2}\frac{e}{M_2}\left(1 - \frac{\Omega_{c2}^2}{\omega^2}\right)^{-1} - n_0\frac{e}{kT_e}\right]\phi \approx 0 \text{ (plasma approximation)}$$

$$1 = (1-\epsilon)\frac{k^2 v_{s1}^2}{\omega^2 - \Omega_{c1}^2} + \epsilon\frac{k^2 v_{s2}^2}{\omega^2 - \Omega_{c2}^2} \quad \Leftarrow$$

(b) There are two roots, one near $\omega = \Omega_{c1}$ and one near $\omega = \Omega_{c2}$., If $\epsilon \to 0$, the root near Ω_{c2} approaches Ω_{c2} to keep the last term finite. The usual root, near Ω_{c1}, is shifted by the presence of the M_2 species:

$$\omega^2 - \Omega_{c1}^2 = k^2 v_{s1}^2 - \epsilon\left[k^2 v_{s1}^2 - k^2 v_{s2}^2 \frac{\omega^2 - \Omega_{c1}^2}{\omega^2 - \Omega_{c2}^2}\right]$$

In the last term, we may approximate ω^2 by $\Omega_{c1}^2 + k^2 v_{s1}^2$. Thus,

$$\omega^2 \approx \Omega_{c1}^2 + k^2 v_{s1}^2 + \epsilon\left[\frac{k^2 v_{s2}^2}{\Omega_{c1}^2 - \Omega_{c2}^2} - 1\right] k^2 v_{s1}^2$$

(c)

$$1 = \frac{1}{2}\frac{k^2 v_{sD}^2}{\omega^2 - \omega_{cD}^2} + \frac{1}{2}\frac{k^2 v_{sT}}{\omega^2 - \Omega_{cT}^2}$$

$$v_{sD}^2 = KT_e/M_D = (10^4)(1.6\times 10^{-19})/(2)(1.67\times 10^{-27}) = 4.79\times 10^{11}$$

$$v_{sT}^2 = \tfrac{2}{3} v_{sD}^2 = 3.19\times 10^{11}$$

$$\Omega_{cD} = eB/M_D = (1.6\times 10^{-19})(5)/(2)(1.67\times 10^{-27}) = 2.40\times 10^8$$

$$\Omega_{cT} = \frac{2}{3}\Omega_{cD} = 1.60\times 10^8 \qquad k = 100\ \mathrm{m}^{-1}$$

$$(\omega^2 - \Omega_{cD}^2)(\omega^2 - \Omega_{cT}^2) = \tfrac{1}{2}k^2[v_{sD}^2(\omega^2 - \Omega_{cT}^2) + v_{sT}^2(\omega^2 - \Omega_{cD}^2)]$$

$$\omega^4 - \omega^2[\Omega_{cD}^2 + \Omega_{cT}^2 + \tfrac{1}{2}k^2(v_{sD}^2 + v_{sT}^2)]$$
$$+ \Omega_{cD}^2\Omega_{cT}^2 + \tfrac{1}{2}k^2(v_{sD}^2\Omega_{cT}^2 + v_{sT}^2\Omega_{cD}^2) = 0$$

$$\omega^4 - \omega^2[8.32\times 10^{16} + 3.99\times 10^{15}] + 1.47\times 10^{33} + 1.53\times 10^{32} = 0$$

$$\omega^4 - 8.72\times 10^{16}\omega^2 + 1.63\times 10^{33} = 0$$

$$\omega^2 = \tfrac{1}{2}[8.72\times 10^{16} \pm (7.60\times 10^{33} - 6.52\times 10^{33})^{1/2}]$$
$$= 6.0\times 10^{16}, \qquad 2.72\times 10^{16}$$

$$\omega = 2.45,\ 1.65\times 10^8\ \mathrm{sec}^{-1} \qquad f = 39 \text{ and } 26.3\ \mathrm{MHz}$$

4-32.

$$\mathscr{E} = n_0\left\langle\frac{1}{2}mv_e^2\right\rangle \qquad v_e = \frac{e}{im\omega}E$$

$$\therefore\ \langle v_e^2\rangle = \frac{e^2}{m^2\omega^2}\langle E^2\rangle$$

$$\mathscr{E} = n_0\frac{1}{2}m\frac{e^2}{m^2\omega^2}\langle E^2\rangle = \frac{\epsilon_0\omega_p^2}{\omega^2}\frac{\langle E^2\rangle}{2}$$

But $\omega^2 = \omega_p^2\ \therefore\ \mathscr{E} = \tfrac{1}{2}\epsilon_0\langle E^2\rangle$.

4-33.

$$\mathscr{E} = n_0\langle\tfrac{1}{2}Mv_i^2\rangle \qquad v_i \approx E_1/B_0$$

$\therefore\ \mathscr{E} = \tfrac{1}{2}Mn_0\langle E_1^2\rangle/B_0$. But $\nabla\times\mathbf{E}_1 = -\dot{B}_1\ \therefore\ \langle E_1^2\rangle = (\omega^2/k^2)\langle B_1^2\rangle$

$$\mathscr{E} = \frac{Mn_0}{2B_0^2}\frac{\omega^2}{k^2}\langle B_1^2\rangle.$$

For Alfvén wave,

$$\frac{\omega^2}{k^2} = \frac{B_0^2}{\mu_0 n_0 M} = \frac{\langle B_1^2 \rangle}{2\mu_0}$$

4-34. (a) With the L-wave, the cutoff occurs at $\omega = \omega_L$, so that one requires $\omega_L^2 < \epsilon\omega^2$. Since $\omega_L < \omega_p$ if n_0 is fixed (Problem 4-15), one can go to higher values of n_0 (for constant $\epsilon\omega^2$) with the L-wave than with the O-wave.

(b) For the L-cutoff,

$$\frac{\omega_p^2}{\omega^2} = 1 + \frac{\omega_c}{\omega} \quad \therefore \; n_c = \frac{\epsilon_0 m\omega^2}{e^2}\left(1 + \frac{\omega_c}{\omega}\right)$$

Thus, to double the usual cutoff density of $\epsilon_0 m\omega^2/e^2$, one must have $f_c = f$

$$f = \frac{c}{\lambda} = \frac{3 \times 10^8}{337 \times 10^{-6}} = 8.9 \times 10^{11}\ \text{Hz}$$

$$f_c = 28 \times 10^9\ \text{Hz/T} \quad \therefore \; B_0 = \frac{8.9 \times 10^{11}}{28 \times 10^9} = 31.8\ \text{T}$$

This would be unreasonably expensive.

(c) The plasma has a density maximum at the center, so it behaves like a convex lens. Such a lens focuses if $\tilde{n} > 1$ and defocuses if $\tilde{n} < 1$. The whistler wave always travels with $v_\phi < c$ (Problem 4-19), so $\tilde{n} = c/v_\phi > 1$, and the plasma focuses this wave.

(d) The question is one of accessibility. If $\omega < \omega_c$ everywhere, the whistler wave will propagate regardless of n_0. However, if $\omega > \omega_c$, the wave will be cut off in regions of low density. From (b) above, we see that a field of 31.8 T is required; this seems too large for the scheme to be practical.

4-35. The answer should come out the same as for cold plasma.

4-36. The linearized equation of motion for either species is

$$-i\omega m n_0 \mathbf{v}_1 = q n_0(\mathbf{E} + \mathbf{v}_1 \times \mathbf{B}_0) - \gamma k T_i \mathbf{k} n_1$$

Thus

$$-i\omega m n_0 \mathbf{k} \cdot \mathbf{v}_1 = q n_0(\mathbf{k} \cdot \mathbf{E} + \mathbf{k} \cdot \mathbf{v}_1 \times \mathbf{B}_0) - \gamma k T_i k^2 n_1.$$

But $\mathbf{k} \cdot \mathbf{E} = 0$ for transverse wave, and $\mathbf{k} \cdot (\mathbf{v}_1 \times \mathbf{B}_0) = -\mathbf{v}_1 \cdot (\mathbf{k} \times \mathbf{B}_0) = 0$ by assumption. The linearized equation of continuity is

$$-i\omega n_1 + n_0 i\mathbf{k} \cdot \mathbf{v}_1 = 0$$

Substituting for $\mathbf{k} \cdot \mathbf{v}_1$, we have

$$i\omega^2 m n_1 = i\gamma k T k^2 n_1$$

Thus n_1 is arbitrary, and we may take it to be 0. Then the ∇p term vanishes for both ions and electrons.

4-44. For a given density, the highest cutoff frequency is ω_R. Thus the lowest bound for n is given by $\omega = \omega_R$.

$$\frac{\omega_p^2}{\omega^2} = \frac{f_p^2}{f^2} = 1 - \frac{\omega_c}{\omega} = 1 - \frac{(1.6 \times 10^{-19})(36 \times 10^{-4})}{(0.91 \times 10^{-30})(2\pi)(1.2 \times 10^8)} = 0.16$$

$$n = f_p^2/q^2 = (0.16)(1.2 \times 10^8)^2 q^{-2} = 2.8 \times 10^{13}\ \mathrm{m}^{-3}$$

4-46. Let $\omega = \omega_R$ at r_1, where $n = n_1$, $\omega_p = \omega_{p1}$; and $\omega = \omega_h$ at r_2, where $n = n_2$, $\omega_p = \omega_{p2}$. Then

$$\omega_{p2}^2 = \omega^2 - \omega_c^2 \qquad \text{[4-105]}$$

$$\omega_{p1}^2 = \omega^2 - \omega\omega_c \qquad \text{[4-107]}$$

Thus

$$\omega_{p2}^2 - \omega_{p1}^2 = \omega_c(\omega - \omega_c) = (n_2 - n_1)e^2/\epsilon_0 m$$

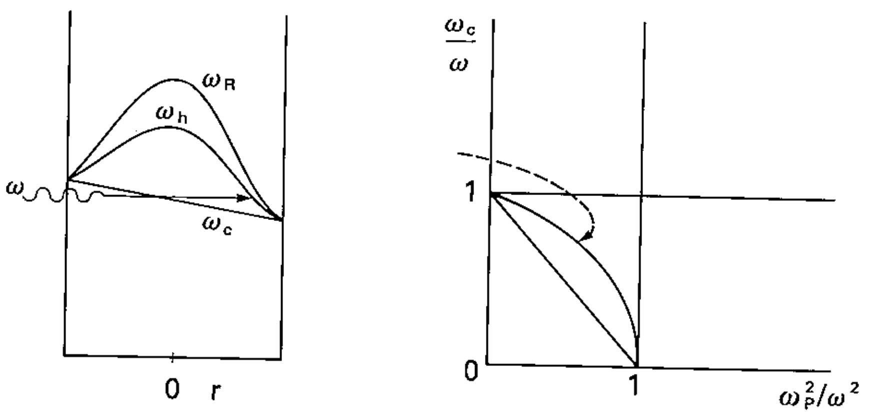
But

$$n_2 - n_1 \approx d|\partial n/\partial r| \approx n_1 d/r_0 = (\epsilon_0 m/e^2)(\omega)(\omega - \omega_c)(d/r_0)$$

So

$$d \approx (\omega_c/\omega) r_0$$

4-47. **(a)** The accessible resonance is on the *far side*, past the density maximum.



(b) Let ω_{c0} be ω_c at the left boundary, and ω_c be the value at the resonance layer, where $\omega = \omega_p$. Then we require

$$\omega_{c0} > \omega, \qquad \text{where } \omega^2 = \omega_c^2 + \omega_p^2$$

Thus

$$\omega_{c0}^2 > \omega_c^2 + \omega_p^2 \qquad \omega_{c0}^2 - \omega_c^2 > \omega_p^2$$

$$(\omega_{c0} + \omega_c)(\omega_{c0} - \omega_c) \approx 2\omega_c \Delta\omega_c > \omega_p^2$$

$$\frac{\Delta\omega_c}{\omega_c} = \frac{\Delta B_0}{B_0} > \frac{\omega_p^2}{2\omega_c^2}$$

4-48. These are the upper and lower hybrid frequencies and right- and left-hand cutoff frequencies with ion motions included. Note that $\omega_p^2/\omega_c = \Omega_p^2/\Omega_c$.
Resonance:

$$\omega^4 - (\omega_p^2 + \omega_c^2 + \Omega_p^2 + \Omega_c^2) + \omega_p^2\Omega_c^2 + \omega_c^2\Omega_p^2 + \omega_c^2\Omega_c^2 = 0$$

$$\omega_+^2 \approx \omega_h^2 + \Omega_p^2(1 - \omega_c^2/\omega_h^2) \qquad \text{(upper hybrid)}$$

$$\omega_-^2 \approx \omega_c^2\Omega_p^2/\omega_h^2 \qquad \text{or} \qquad \frac{1}{\omega_-^2} = \frac{1}{\omega_c\Omega_c} + \frac{1}{\Omega_p^2} \qquad \text{(lower hybrid)}$$

Cutoff:

$$\frac{\bar{\omega}_p^2}{\omega^2} = \left(1 \mp \frac{\omega_c}{\omega}\right)\left(1 \pm \frac{\Omega_c}{\omega}\right) \qquad \left(\begin{matrix} R \\ L \end{matrix}\text{cutoff}\right)$$

This is more easily obtained, without approximation, from the form given in Problem 4-50.

5-1. (a) $D_e = KT_e/m\nu$

$$\sigma = (6\pi)(0.53 \times 10^{-10})^2 = 5.29 \times 10^{-20}\ \text{m}^2$$

$$v = \left(\frac{2E}{m}\right)^{1/2} = \left[\frac{(2)(2)(1.6 \times 10^{-19})}{(9.11 \times 10^{-31})}\right]$$

$$= 8.39 \times 10^5\ \text{m/sec}$$

From Problem 1-1b,

$$n_0 = (3.3 \times 10^{19})(10^3) = 3.3 \times 10^{22}\ \text{m}^{-3}$$

$$\nu = n_0\overline{\sigma v} = n_0\sigma v = (3.3 \times 10^{22})(5.29 \times 10^{-20})(8.39 \times 10^5)$$

$$= 1.46 \times 10^9\ \text{sec}^{-1}$$

$$D_e = \frac{(2)(1.6 \times 10^{-19})}{(9.11 \times 10^{-31})(1.46 \times 10^9)} = 2.4 \times 10^2\ \text{m}^2/\text{sec}$$

(b) $j = \mu neE$

$$\mu_e = eD_e/KT_e = \frac{(1.6 \times 10^{-19})(2.4 \times 10^2)}{(2)(1.6 \times 10^{-19})}$$

$$= 1.2 \times 10^2\ \text{m}^2/\text{V sec}$$

$$E = \frac{j}{\mu ne} = \frac{2 \times 10^3}{(1.2 \times 10^2)(10^{16})(1.6 \times 10^{-19})} = 1.04 \times 10^4\ \text{V/m}$$

5-2.

$$\frac{\partial n}{\partial t} = D\nabla^2 n - \alpha n^2$$

$$D\nabla^2 n = D\frac{\partial^2 n}{\partial x^2} = -Dn_0\left(\frac{\pi}{2L}\right)^2 \cos\frac{\pi x}{2L} = -D\left(\frac{\pi}{2L}\right)^2 n = -\alpha n^2$$

$$\therefore\ n = \frac{D}{\alpha}\left(\frac{\pi}{2L}\right)^2 = \frac{0.4}{10^{-15}}\left(\frac{\pi}{0.06}\right)^2 = 1.1\times 10^{18}\ \mathrm{m}^{-3}$$

5-4. (a) From Problem 5-1a, $\nu_{en} = 1.46\times 10^9\ \sec^{-1}$. We need to find whether $\mu_{e\perp}/\mu_{i\perp}$ is large or small:

$$\frac{\mu_e}{\mu_i} = \frac{M\nu_{in}}{m\nu_{en}} \qquad \nu_{jn} = n_n\sigma v_j \propto v_{\mathrm{th}j} \propto m_j^{-1/2}$$

since σ is approximately the same for ion–neutral and electron–neutral collisions. Thus

$$\frac{\mu_e}{\mu_i} \approx \left(\frac{M}{m}\right)^{1/2} = (4\times 1{,}836)^{1/2} = 85.7$$

$$\omega_c = \frac{eB}{m} = \frac{(1.6\times 10^{-19})(0.2)}{9.11\times 10^{-31}} = 3.52\times 10^{10}$$

$$\omega_c\tau_{en} = \frac{3.52\times 10^{10}}{1.46\times 10^9}\times 24 \qquad 1+\omega_c^2\tau_{en}^2 = 580$$

$$\Omega_c\tau_{in} = \omega_c\tau_{en}\left(\frac{m}{M}\right)\left(\frac{M}{m}\right)^{1/2} = (24)(85.7)^{-1} = 0.28$$

$$\frac{\mu_{e\perp}}{\mu_{i\perp}} = \frac{\mu_e}{\mu_i}\,\frac{1+\Omega_c^2\tau_{in}^2}{1+\omega_c^2\tau_{en}^2} = (85.7)\frac{1.08}{580} = 0.16 \ll 1$$

$$\therefore\ D_{a\perp} = \frac{\mu_{i\perp}D_{e\perp}+\mu_{e\perp}D_{i\perp}}{\mu_{i\perp}+\mu_{e\perp}} \approx D_{e\perp} + \frac{\mu_{e\perp}}{m_{i\perp}}D_{i\perp}$$

$$= D_{e\perp} + 0.16D_{i\perp}$$

But

$$D = \frac{KT}{e}\mu$$

$$\therefore\ \frac{D_{i\perp}}{D_{e\perp}} = \frac{\mu_{i\perp}}{\mu_{e\perp}}\,\frac{T_i}{T_e} = \frac{1}{0.16}\,\frac{0.1}{2} = 0.3$$

$$\therefore\ D_{a\perp} = D_{e\perp}[1+(0.16)(0.3)] = 1.05D_{e\perp} \approx D_{e\perp}$$

l whether

collisions.

(b)

$$\frac{a}{(D\tau)^{1/2}} = 2.4 \;\therefore\; \tau = \left(\frac{a}{2.4}\right)^2 \frac{1}{D_{a\perp}}$$

$$\tau = \frac{1}{(2.4 \times 10^{-2})^2} \frac{1}{D_{e\perp}}$$

$$D_{e\perp} = \frac{2.4 \times 10^2}{580} = 0.4140 \text{ (from Problem 5-1)}$$

$$\therefore\; \tau = 42\ \mu\text{sec}$$

5-5.

$$\Gamma = -D\, dn/dx \qquad n = n_0(1 - x/L)$$

$$\Gamma = Dn_0/L \qquad (x > 0)$$

$$Q = 2\Gamma = 2Dn_0/L \;\therefore\; n_0 = QL/2D$$

5-7.

$$\lambda_{ei} \approx v_{\text{th}e}\tau_{ei} = v_{\text{th}e}/\nu_{ei}$$

But $v_{\text{th}e} \propto T_e^{1/2}$ and $\nu_{ei} \propto T_e^{-3/2}$

$$\therefore\; \lambda_{ei} \propto T_e^{1/2}/T_e^{-3/2} \propto T_e^2$$

5-8.

$$\eta_{\parallel} = 5.2 \times 10^{-5} \frac{\ln \Lambda}{T_{ev}^{3/2}} \Omega\text{-m} \qquad (\text{assume } Z = 1)$$

$$= \frac{(5.2 \times 10^{-5})(10)}{(500)^{3/2}} = 4.65 \times 10^{-8}\ \Omega\text{-m}$$

$$j = I/A = (2 \times 10^5)/(7.5 \times 10^{-3}) = 2.67 \times 10^7\ \text{A/m}^2$$

$$E = \eta_{\parallel} j = (4.65 \times 10^{-8})(2.67 \times 10^7) = 1.2\ \text{V/m}$$

5-9. (a)

$$KT_i = 20\ \text{keV} \qquad KT_e = 10\ \text{keV} \qquad n = 10^{12}\ \text{m}^{-3}$$

$$B = 5T \qquad D_{\perp} = \frac{\eta n(KT_i + KT_e)}{B^2}$$

$$\eta_{\perp} = (2.0)(5.2 \times 10^{-5}) \frac{\ln \Lambda}{T_{ev}^{3/2}} = \frac{(10^{-3})(10)}{(10^4)^{3/2}}$$

$$= 1.0 \times 10^{-9}\ \Omega\text{-m}$$

$$D_{\perp} = \frac{(1.0 \times 10^{-9})(10^{21})(3 \times 10^4)(1.6 \times 10^{-19})}{5^2}$$

$$= 3.0 \times 10^{-4}\ \text{m}^2/\text{sec}$$

(b)

$$\frac{dN}{dt} = 2\pi r L \Gamma_r \qquad \Gamma_r = -D_\perp \frac{\partial n}{\partial r}$$

$$\frac{\partial n}{\partial r} = \frac{n}{0.1} \qquad r = 0.50 \text{ m} \qquad L = 100 \text{ m}$$

$$-\frac{dN}{dt} = (2\pi)(0.50)(10^2)(2.0 \times 10^{-4})(10^{21}/0.10) = 6 \times 10^{20} \text{ sec}^{-1}$$

(c)

$$\tau = \frac{N}{-dN/dt} = \frac{n\pi r^2 L}{-dN/dt} \qquad r_{\text{effective}} = 0.55 \text{ m}$$

$$\tau = \frac{(10^{21})(\pi)(0.55)^2(10^2)}{6 \times 10^{20}} = 150 \text{ sec}$$

5-13.

$$\eta_\parallel = 5.2 \times 10^{-5} \frac{\ln \Lambda}{T_{ev}^{3/2}} \Omega\text{-m} = (5.2 \times 10^{-5}) \frac{10}{10^{3/2}}$$

$$= 1.6 \times 10^{-5} \ \Omega\text{-m}$$

$$\eta j^2 = (1.6 \times 10^{-5})(10^5)^2 = 1.6 \times 10^5 \text{ W/m}^3$$

$$= 1.6 \times 10^5 \text{ J/(m}^3\text{-sec)}$$

$$= (1.6 \times 10^5)/(1.6 \times 10^{-19}) = 10^{24} \text{ eV/m}^3\text{-sec}$$

$$= \frac{dE_{ev}}{dt}$$

$$E = \frac{3}{2} n K T_e \ \therefore \ \frac{dE_{ev}}{dt} = \frac{3}{2} n \frac{dT_{ev}}{dt}$$

$$\frac{dT_{ev}}{dt} = \frac{2}{3} \frac{1}{10^{19}} 10^{24} = 0.67 \times 10^5 \text{ eV/sec} = 0.067 \text{ eV/}\mu\text{sec}$$

5-15. (a)

$$en(\overset{0}{\cancel{E_\theta}} - v_{ir}B) - \overset{0}{\cancel{\nabla_\theta}} p_i - e^2 n^2 \eta (v_{i\theta} - v_{e\theta}) = 0$$

$$-en(\overset{0}{\cancel{E_\theta}} - v_{er}B) - \overset{0}{\cancel{\nabla_\theta}} p_e + e^2 n^2 \eta (v_{i\theta} - v_{e\theta}) = 0$$

add:

$$-v_{ir}B + v_{er}B = 0 \ \therefore \ v_{ir} = v_{er}$$

(This shows ambipolar diffusion.)

(b)

$$en(E_r + v_{i\theta}B) - \frac{\partial p_i}{\partial r} - e^2n^2\eta(\cancel{v_{ir}}^{\,0} - v_{er}) = 0$$

$$-en(E_r + v_{e\theta}B) - \frac{\partial p_e}{\partial r} + e^2n^2\eta(v_{ir} - v_{er}) = 0$$

$$v_{i\theta} = -\frac{E_r}{B} + \frac{1}{enB}\frac{\partial p_i}{\partial r} = v_E + v_{Di}$$

$$v_{e\theta} = -\frac{E_r}{B} - \frac{1}{enB}\frac{\partial p_e}{\partial r} = v_E + v_{De}$$

(c) From the first equation in (a),

$$v_{ir} = -\frac{e^2n^2\eta}{enB}(v_{i\theta} - v_{e\theta})$$

$$= \frac{en\eta}{B}\frac{1}{enB}\left(\frac{\partial p_i}{\partial r} + \frac{\partial p_e}{\partial r}\right) = -\frac{\eta}{B^2}\frac{\partial p}{\partial r} = v_{er}$$

(This shows the absence of cross-field mobility.)

5-17. (a)

$$\rho_0\frac{\partial \mathbf{v}_1}{\partial t} = \mathbf{j}_1 \times \mathbf{B}_0 \qquad (1)$$

$$\mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_0 = \eta\mathbf{j}_1 \qquad (2)$$

$$\nabla \times \mathbf{E}_1 = -\dot{\mathbf{B}}_1 \qquad \nabla \times \mathbf{B}_1 = \mu_0\mathbf{j}_1$$

$$\nabla \times \nabla \times \mathbf{E}_1 = -\nabla \times \dot{\mathbf{B}}_1 = -\mu_0\dot{\mathbf{j}}_1$$

$$-\mathbf{k}(\cancel{\mathbf{k}\cdot}^{\,0}\mathbf{E}) + k^2\mathbf{E}_1 = i\omega\mu_0\mathbf{j}_1 \qquad (3)$$

$$\mathbf{k}\cdot\mathbf{E} = 0 \qquad \text{(transverse wave)}$$

Solve for $\mathbf{v}_1$ in (2):

$$\mathbf{E}_1 \times \mathbf{B}_0 + \underbrace{(\mathbf{v}_1 \times \mathbf{B}_0) \times \mathbf{B}_0}_{-\mathbf{v}_{1\perp}B_0^2} = \eta\mathbf{j}_1 \times \mathbf{B}_0$$

$$v_{1\perp} = \frac{\mathbf{E}_1 \times \mathbf{B}_0}{B_0^2} - \frac{\eta\mathbf{j}_1 \times \mathbf{B}_0}{B_0^2}$$

Substitute in (1), which has no parallel component anyway:

$$-i\omega\rho_0\left(\frac{\mathbf{E}_1 \times \mathbf{B}_0}{B_0^2} - \frac{\eta\mathbf{j}_1 \times \mathbf{B}_0}{B_0^2}\right) = \mathbf{j}_1 \times \mathbf{B}_0$$

Since, by Eq. (3), $\mathbf{E}$ and $\mathbf{j}_1$ are in the same direction, take them both to be in the $\hat{\mathbf{x}}$-direction. Then the y-component is

$$\frac{E_1}{B_0} = \left(\frac{iB_0}{\omega\rho_0} + \frac{\eta}{B_0}\right)j_1$$

Equation (3) becomes

$$k^2 E_1 = \mu_0 i\omega \frac{E_1}{B_0}\left(\frac{iB_0}{\omega\rho_0} + \frac{\eta}{B_0}\right)^{-1}$$

$$= \mu_0 \omega^2 \left(\frac{B_0^2}{\rho_0} - i\eta\omega\right)^{-1} E_1$$

$$\frac{\omega^2}{k^2} = \mu_0\left(\frac{B_0^2}{\rho_0} - i\omega\eta\right)$$

(b)

$$k = (\mu_0\omega^2)^{1/2}\left(\frac{B_0^2}{\rho_0} - i\omega\eta\right)^{-1/2}$$

$$= \omega\left(\frac{\mu_0\rho_0}{B_0^2}\right)^{1/2}\left(1 - \frac{i\omega\eta\rho_0}{B_0^2}\right)^{-1/2}$$

$$\operatorname{Im}(k) = \omega\frac{\omega\eta\rho_0}{2B_0^2}\left(\frac{\mu_0\rho_0}{B_0^2}\right)^{1/2} = \frac{\omega^2\eta}{2}\frac{1}{v_A^3}$$

But for small η, $\omega \approx kv_A$, where $k = \operatorname{Re}(k)$

$$\therefore\ \operatorname{Im}(k) \approx \frac{(\eta)(k^2)}{2v_A}$$

6-4. (a)

$$\mathbf{j}\times\mathbf{B} = \nabla p = KT\nabla n \qquad (KT = KT_e + KT_i \text{ here})$$

$$(\mathbf{j}\times\mathbf{B})\times\mathbf{B} = KT\nabla n \times \mathbf{B} = \mathbf{B}(\mathbf{j}\cdot\mathbf{B}) - \mathbf{j}B^2$$

The parallel component is $0 = j_\| B^2 - j_\| B^2$ $\therefore$ $j_\|$ is arbitrary. The perpendicular component is

$$\mathbf{j}_\perp = \frac{KT}{B^2}\mathbf{B}\times\nabla n = \frac{KT}{B}\frac{\partial n}{\partial r}\hat{\theta}$$

(b)

$$\int \nabla\times\mathbf{B}\cdot d\mathbf{S} = \mu_0\int \mathbf{j}\cdot d\mathbf{S}$$

$$\oint \mathbf{B}\cdot d\mathbf{L} = \mu_0\int \mathbf{j}\cdot d\mathbf{S} = \mu_0 L\int_0^\infty j_\theta\, dr$$

since $\mathbf{j}$ and $d\mathbf{S}$ are both in the $\hat{\theta}$ direction, and L is the width of the loop in the $\hat{\mathbf{z}}$ direction. By symmetry, there can be no B_r, so only the two z-legs of the loop contribute to the line integral. Substituting for j_θ, we have

$$(B_{ax} - B_0)L = \mu_0 LKT\int_0^\infty \frac{\partial n/\partial r}{B(r)}\, dr$$

(c) $\partial n/\partial r = -n_0\,\delta(r-a)$, since $\partial n/\partial r$ is a function that is zero everywhere except at $r = a$, is $-\infty$ there, and has an integral equal to $-n_0$. Thus

$$B_{ax} - B_0 = \mu_0 KT\int_0^\infty - n_0\frac{\delta(r-a)}{B(r)}\, dr$$

Since all the diamagnetic current is concentrated at $r = a$, B takes a jump from a constant value B_{ax} inside the plasma to another constant value B_0 outside. (Remember that the field inside an infinite solenoid is uniform.) Upon integrating across the jump, one obtains the average value of B on the two sides, i.e., $B(a) = \frac{1}{2}(B_{ax} + B_0)$. Thus

$$B_{ax} - B_0 = \mu_0 KTn_0 \frac{-1}{\frac{1}{2}(B_{ax} + B_0)}$$

$$B_{ax}^2 - B_0^2 = -2\mu_0 n_0 KT$$

$$1 - \frac{B_{ax}^2}{B_0^2} = \frac{2\mu_0 n_0 KT}{B^2} \equiv \beta = 1 \;\therefore\; B_{ax} = 0$$

6-5. **(a)** By Faraday's law, $V = -d\Phi/dt$

$$\therefore \int V\,dt = -N \int \frac{d\Phi}{dt}\,dt = -N\Delta\Phi$$

Since $\Delta\Phi$ is the flux change due to the diamagnetic decrease in B,

$$-N\Delta\Phi = -N \int (\mathbf{B} - \mathbf{B}_0) \cdot d\mathbf{S}$$

The sign depends on which side of V is considered positive. In practice, this is of no consequence because the oscilloscope trace can easily be inverted by using the polarity switch.

(b) In Problem 6-4b, we can draw the loop so that its inner leg lies at an arbitrary radius r rather than on the axis. We then have

$$B(r) - B_0 = \mu_0 KT \int_r^\infty \frac{\partial n/\partial r}{B(r')}\,dr' \approx \mu_0 KT \int_r^\infty \frac{\partial n/\partial r'}{B_0}\,dr'$$

where again KT is short for $\sum KT$

$$\frac{\partial n}{\partial r} = n_0 \left(\frac{-2r}{r_0^2}\right) e^{-r^2/r_0^2}$$

$$B(r) - B_0 = \frac{\mu_0 KT}{B_0} \frac{n_0}{r_0^2} \int_\infty^r e^{-r'^2/r_0^2} 2r'\,dr'$$

$$= \frac{\mu_0 n_0 KT}{B_0} [e^{-r'^2/r_0^2}]_r^\infty = \frac{-\mu_0 n_0 KT}{B_0} e^{-r^2/r_0^2}$$

This is the diamagnetic change in B at any r. To get the loop signal, we must integrate over the plasma cross section.

$$\int V\,dt = -N \int (\mathbf{B} - \mathbf{B}_0) \cdot d\mathbf{S} = -N \iint [B(r) - B_0] r\,dr\,d\theta$$

where both $\mathbf{B}$ and $d\mathbf{S}$ are in the $\hat{\mathbf{z}}$ direction. Substituting for $B(r) - B_0$ and assuming the coil lies well outside the plasma, we have

$$\int V\,dt = N \frac{\mu_0 n_0 KT}{B_0} 2\pi \int_0^\infty e^{-r^2/r_0^2} r\,dr$$

$$= N\pi \frac{\mu_0 n_0 KT}{B_0} r_0^2 [e^{-r^2/r_0^2}]_\infty^0 = \frac{1}{2} N\pi r_0^2 \left(\frac{2\mu_0 n_0 KT}{B_0^2}\right) B_0$$

(c) The quantity in parentheses is β by definition; hence,

$$\int V\,dt = \frac{1}{2}N\pi r_0^2\beta B_0$$

Both sides of this equation have units of flux.

6-6. (a) For each stream, we have

$$m\left(\frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_1\right) = -e\mathbf{E}_1 = (-i\omega + ikv_0)\mathbf{v}_1$$

$$\mathbf{v}_1 = \frac{-ie\mathbf{E}_1}{m(\omega - kv_0)}$$

$$\frac{\partial n_1}{\partial t} + n_0(\nabla \cdot \mathbf{v}_1) + (\mathbf{v}_0 \cdot \nabla)n_1 = 0$$

$$(-i\omega + ikv_0)n_1 + ikn_0v_1 = 0 \qquad n_1 = n_0\frac{kv_1}{\omega - kv_0}$$

$$\therefore\ n_{1j} = n_{0j}\frac{-ikE_1e}{m(\omega - kv_{0j})^2}$$

Poisson: $ikE_1 = (e/\epsilon_0)(n_{1a} + n_{1b})$, where stream a has $v_{0a} = v_0\hat{\mathbf{x}}$, $n_{0a} = \frac{1}{2}n_0$; stream b has $v_{0b} = -v_0\hat{\mathbf{x}}$, $n_{0b} = \frac{1}{2}n_0$. Thus

$$ikE_1 = -\left(\frac{e}{\epsilon_0}\right)\left(\frac{-ikeE_1}{m}\right)\left[\frac{\frac{1}{2}n_0}{(\omega - kv_0)^2} + \frac{\frac{1}{2}n_0}{(\omega + kv_0)^2}\right]$$

$$1 = \frac{n_0e^2}{\epsilon_0 m}\cdot\frac{1}{2}\left[\frac{1}{(\omega - kv_0)^2} + \frac{1}{(\omega + kv_0^2)}\right]$$

$$1 = \frac{1}{2}\omega_p^2\left[\frac{1}{(\omega - kv_0)^2} + \frac{1}{(\omega + kv_0)^2}\right]$$

(b)

$$1 = \omega_p^2\frac{\omega^2 + k^2v_0^2}{(\omega^2 - k^2v_0^2)^2}$$

$$\omega^4 - (\omega_p^2 + 2k^2v_0^2)\omega^2 + k^2v_0^2(k^2v_0^2 - \omega_p^2) = 0$$

$$\omega^2 = \tfrac{1}{2}(\omega_p^2 + 2k^2v_0^2) \pm \tfrac{1}{2}(\omega_p^4 + 8\omega_p^2k^2v_0^2)^{1/2}$$

Let

$$x = \frac{2k^2v_0^2}{\omega_p^2} \qquad y^2 = \frac{2\omega^2}{\omega_p^2}$$

Then

$$y^2 = 1 + x \pm (1 + 4x)^{1/2}$$

y can be complex only if the (−) sign is taken. Then y is pure imaginary, and we can let $y = i\gamma$:

$$\gamma^2 = (1+4x)^{1/2} - (1+x)$$

$$\frac{d}{dx}(\gamma^2) = 2(1+4x)^{-1/2} - 1 = 0 \qquad x = \frac{3}{4}$$

Thus

$$\gamma^2 = (1+3)^{1/2} - \tfrac{7}{4} = \tfrac{1}{4}$$

$$\gamma = \frac{1}{2} = \frac{\sqrt{2}\,\mathrm{Im}(\omega)}{\omega_p} \qquad \mathrm{Im}(\omega) = \frac{\omega_p}{2^{3/2}}$$

6-8. (a)

$$1 = \omega_p^2\left[\frac{1}{\omega^2} + \frac{\delta}{(\omega - ku)^2}\right]$$

where $\omega_p^2 \equiv n_0 e^2/\epsilon_0 m$.

(b) This equation is the same as Eq. [6-30] except that m/M is replaced by δ, which is also small, and that the rest frame has changed to one moving with velocity u. The maximum growth rate does not depend on frame, as can be seen from Fig. 6-11 by imagining γ to be plotted in the z direction vs. x and y; a shift in the origin of x will not affect the peak. Analogy with Eq. [6-35] then gives

$$\gamma_{\max} \approx \delta^{1/3}\omega_p$$

(The exact constant that should appear here is $3^{1/2}2^{-4/3} = 0.69$. The derivation of $\gamma_{\max}$, which is difficult because the dispersion relation is cubic, and the proof that it is independent of frame for real k are left as exercises for the advanced student.)

6-9. (a) Since only the y component of $\mathbf{v}_i$ and $\mathbf{E}$ are involved, the given relation is easily found from Eqs. [4-98(b)] and [6-23], plus continuity and Poisson's equation. Note that Ω_p is defined with n_0, not $(1/2)n_0$.

(b) Let $\alpha \equiv \frac{1}{2}\Omega_p^2(1 + \omega_p^2/\omega_c^2)^{-1}$, $\beta \equiv k^2 v_0^2$. Then the dispersion relation reduces to

$$\omega^4 - 2(\alpha+\beta)\omega^2 + \beta^2 - 2\alpha\beta = 0$$

The dispersion $\omega(k)$ is given by

$$\omega^2 = \alpha + \beta \pm (\alpha^2 + 4\alpha\beta)^{1/2}$$

Instability occurs if $(\alpha^2 + 4\alpha\beta)^{1/2} > \alpha + \beta$, or $\beta < 2\alpha$, i.e.,

$$k^2 < (\Omega_p^2/v_0^2)(1 + \omega_p^2/\omega_c^2)^{-1}$$

When this is satisfied, the growth rate is given by

$$\gamma = [(\alpha^2 + 4\alpha\beta)^{1/2} - (\alpha + \beta)]^{1/2}$$

; stream

7-3. **(a)**

$$f_p(v) = \frac{n_p}{a\pi^{1/2}} e^{-v^2/a^2}$$

$$f_b(v) = \frac{n_b}{b\pi^{1/2}} e^{-(v-V)^2/b^2}$$

(b)

$$f_b'(v) = \frac{n_b}{b\pi^{1/2}} \frac{-2(v-V)}{b^2} e^{-(v-V)^2/b^2}$$

$$f_b''(v) = \frac{-2n_b}{b^3\pi^{1/2}}\left[1 - \frac{2(v-V)^2}{b^2}\right] e^{-(v-V)^2/b^2} = 0$$

$$v - V = \pm b/\sqrt{2} \qquad v_\phi = V - b/\sqrt{2}$$

$$f_b'(v_\phi) = \left(\frac{2}{\pi}\right)^{1/2} \frac{n_b}{b^2} e^{-1/2}$$

(c)

$$f_p'(v_\phi) = \frac{n_p}{a\pi^{1/2}}\left(\frac{-2}{a^2}\right)\left(V - \frac{b}{2^{1/2}}\right) e^{-(V-b/\sqrt{2})^2/a^2}$$

$$\approx -\frac{2n_p V}{a^3\pi^{1/2}} e^{-V^2/a^2} \qquad V \gg b$$

(d)

$$\left(\frac{2}{\pi}\right)^{1/2} \frac{n_b}{b^2} e^{-1/2} = \frac{2n_p V}{a^3\pi^{1/2}} e^{-V^2/a^2}$$

$$\frac{n_b}{n_p} = (2e)^{1/2}\frac{b^2}{a^3} V e^{-V^2/a^2} \qquad \frac{b^2}{a^2} = \frac{T_b}{T_p}$$

$$\therefore \quad \frac{n_b}{n} = (2e^{1/2})\frac{T_b}{T_p}\frac{V}{a} e^{-V^2/a^2}$$

7-8. From Eq. [7-127], we obtain $\sum \alpha_j Z'(\zeta_j) = 2T_i/T_e$, where $\alpha_j = n_{0j}/n_{0e}$, $\zeta_j = \omega/kv_{\text{th}j}$. Assume at first that α_H is small, so that $\alpha_A \approx 1$, $\alpha_H = \alpha$; furthermore, small α means that v_ϕ will be nearly unchanged from v_s of argon. Then doubling the Landau damping rate means $\operatorname{Im} Z'(\zeta_H) = \operatorname{Im} Z'(\zeta_A)$, where $\operatorname{Im} Z'(\zeta_j) = -2i\sqrt{\pi}\zeta_j e^{-\zeta_j^2}$. Thus

$$\zeta_A e^{-\zeta_A^2} = \alpha\zeta_H e^{-\zeta_H^2} \qquad \alpha = \frac{\zeta_A}{\zeta_H} e^{-(\zeta_A^2 - \zeta_H^2)}$$

$$\frac{\zeta_A}{\zeta_H} = \left(\frac{M_A}{M_H}\right)^{1/2} \qquad \alpha = (40)^{1/2} e^{-\zeta_A^2(1-1/40)}$$

$$\zeta_A^2 = \frac{KT_e + 3KT_i}{M_A} \cdot \frac{M_A}{2KT_i} = \frac{13}{2}$$

$$\alpha = \sqrt{40}\, e^{-6.5(0.975)} = 1.12 \times 10^{-2} \approx 1\%$$

Thus α is so small that our initial assumptions are justified.

7-9. (a)

$$\frac{2k^2}{k_{\mathrm{Di}}^2} = Z'(\zeta_i) + \frac{1-\alpha}{\theta_e} Z'(\zeta_e) + \frac{\alpha}{\theta_h} Z'(\zeta_h)$$

(b)

$$Z'(\zeta) \approx -2 - 2i\sqrt{\pi}\zeta\, e^{-\zeta^2}$$

Since $\zeta_h \ll \zeta_e \ll 1$,

$$|\operatorname{Im} Z'(\zeta_h)| \ll |\operatorname{Im} Z'(\zeta_e)|$$

(c) Since $Z'(\zeta_h) \approx Z'(\zeta_e) \approx -2$), the ζ_h term in (a) is negligible compared with the ζ_e term if $\theta_h \gg \theta_e$ and $\alpha < 1/2$. Now the dispersion relation is

$$Z'(\zeta_i) = \frac{2k^2}{k_{\mathrm{Di}}^2} + \frac{2(1-\alpha)}{\theta_e} = \frac{2T_i}{T_e}\left(1 - \alpha + \frac{T_e k^2}{T_i k_{\mathrm{Di}}^2}\right)$$

The last term is $\approx k^2\lambda_D^2$ and is negligible when quasineutraility holds. Thus the ion wave dispersion relation is the same as usual, except that T_i/T_e has been replaced by $(1-\alpha)T_i/T_e$. Since small T_i/T_e means less Landau damping, the hot electrons have decreased ion Landau damping.

8-3. Refer to Fig. 8-4. Take a number of ions with $v = u_0$ and split them into two groups, one with $v = u_0 + \Delta$ and one with $v = u_0 - \Delta$. After acceleration in a potential ϕ, the faster half will have less fractional energy gain (because it started with more energy) and, hence, will have less fractional density decrease. The opposite is true for the slower half, and to first order the total density decrease is the same as if all ions had $v = u_0$. However, there is a second-order effect which makes the slower group dominate. This can be seen by making Δ so large that $v \approx 0$ for the slower half, which clearly must then suffer a huge density decrease. To compensate for this, u_0 must be *increased* to higher than the Bohm value.

8-4. The maximum current occurs when the space charge of decelerated ions near grid 3 decreases the electric field to zero. Thus we can apply the Child-Langmuir law to the region between grids 2 and 3.

$$J = \frac{4}{9}\left[\frac{(2)(1.6\times10^{-19})}{(4)(1.67\times10^{-27})}\right]^{1/2} \frac{(8.85\times10^{-12})(100)^{3/2}}{(10^{-3})^2} = 27.2\,\frac{\mathrm{A}}{\mathrm{m}^2}$$

$$A = \frac{\pi}{4}(4\times10^{-3})^2 = 1.26\times10^{-5}\ \mathrm{m}^2$$

$$I = JA = \underline{0.34\ \mathrm{mA}}$$

8-6. (a) At $\omega_p = \omega$,

$$F_{\mathrm{NL}} = -\frac{\epsilon_0\langle E^2\rangle}{2L} = -\nabla p_{\mathrm{eff}} = \frac{p_{\mathrm{eff}}}{L}$$

$A =$

$$= -\frac{(2\pi)(3\times 10^8)}{(1.06\times 10^{-6})^2}(21.9\times 10^{-10})$$

$$= 3.67\times 10^{12}$$

$$1+\frac{3}{\theta} = \left(\frac{3.67\times 10^{12}}{2.61\times 10^{12}}\right)^2 = 2 \qquad \theta = \frac{T_e}{T_i} = 3 \;\therefore\; T_i = \frac{1}{3}\,\text{keV}$$

8-10. (a)

$$\langle E_0^2\rangle = \frac{1}{2}\bar{E}^2 = \frac{8\omega_1\omega_2\Gamma_1\Gamma_2}{c_1c_2}$$

$$c_1c_2 = \frac{\epsilon_0 k_1^2\omega_p^4}{n_0\omega_0^2 M} \qquad \Gamma_2 = \frac{\omega_p^2}{\omega_2^2}\frac{\nu}{2}$$

$$\langle E_0^2\rangle = \frac{4\omega_1\Gamma_1\omega_0^2\nu}{\omega_2 k_1^2}\frac{n_0 M}{\epsilon_0\omega_p^2} = \frac{4\omega_1\Gamma_1\omega_0^2\nu M m}{\omega_2 k_1^2 e^2}$$

$$\langle v_0^2\rangle = \frac{e^2\langle E_0^2\rangle}{m^2\omega_0^2} = \frac{4\omega_1\Gamma_1\nu M}{\omega_2 k_1^2 m}$$

$$k_1^2 = \frac{\omega_1^2}{v_s^2} = \frac{\omega_1^2 M}{KT_e} = \frac{\omega_1^2 v_e^2 M}{m} \;\therefore\; \frac{\langle v_0^2\rangle}{v_e^2} = \frac{4\Gamma_1\nu}{\omega_1\omega_2}$$

(b)

$$\frac{\langle v_0^2\rangle}{v_e^2} = \frac{4\Gamma_1\nu_{ei}}{\omega_1\omega_0}$$

since $\omega_2 \approx \omega_0$ when $n \ll n_c$.

$$\omega_0 = \frac{2\pi c}{\lambda_0} = \frac{(2\pi)(3\times 10^8)}{10.6\times 10^{-6}} = 1.78\times 10^{14}\ \text{sec}^{-1}$$

$$v_e^2 = \frac{KT_e}{m} = \frac{(10^2)(1.6\times 10^{-19})}{(0.91\times 10^{-30})} = 1.76\times 10^{13}\ \frac{\text{m}^2}{\text{sec}^2}$$

$$\frac{\Gamma_1}{\omega_1} = \left(\frac{\pi}{8}\right)^{1/2}\theta(3+\theta)^{1/2}e^{-(3+\theta)/2} \qquad \theta = \frac{T_e}{T_i} = 10$$

$$= 3.40\times 10^{-2}$$

$$\eta = 5.2\times 10^{-5}\frac{\ln\Lambda}{T_{eV}^{3/2}} = \frac{(5.2\times 10^{-5})(10)}{(100)^{3/2}} = 5.2\times 10^{-7}\ \Omega\text{-m}$$

$$\nu_{ei} = \frac{ne^2\eta}{m} = \frac{(10^{23})(1.6\times 10^{-19})^2(5.2\times 10^{-7})}{(0.91\times 10^{-30})} = 1.46\times 10^9\ \text{sec}^{-1}$$

$$\langle v_0^2\rangle = \frac{(4)(3.4\times 10^{-2})(1.46\times 10^9)}{1.78\times 10^{14}}(1.76\times 10^{13}) = 1.96\times 10^7\ \frac{\text{m}^2}{\text{sec}^2}$$

$\therefore\ p_{\text{eff}} = \frac{1}{2}\epsilon_0\langle E^2\rangle$. But $I_0 = c\epsilon_0\langle E^2\rangle = P/A$, where $P = 10^{12}$ and $A = (\pi/4)(50 \times 10^{-6})^2 = 1.96 \times 10^{-9}\ \text{m}^2$

$$p_{\text{eff}} = \frac{P}{2cA} = \frac{10^{12}}{(2)(3 \times 10^8)(1.96 \times 10^{-9})} = 8.50 \times 10^{11}\,\frac{\text{N}}{\text{m}^2}$$

$$= \frac{(8.50 \times 10^{11})(0.2248)}{(39.37)^2} = 1.23 \times 10^8\,\frac{\text{lb}}{\text{in.}^2}$$

(b)

$$F = pA \qquad P/2c = 10^{12}/(2)(3 \times 10^8) = 1667\ \text{N}$$

$$F = Mg \qquad M = F/g = 1667/9.8 = 170\ \text{kg} = 0.17\ \text{tonnes}$$

(c)

$$2nKT = p_{\text{eff}}$$

$$\therefore\ n = \frac{8.5 \times 10^{11}}{(2)(10^3)(1.6 \times 10^{-19})} = 2.66 \times 10^{27}\ \text{m}^{-3}$$

8-7.

$$F_{\text{NL}} = \nabla p\ \ \therefore\ \frac{\partial}{\partial r}(nKT) = -\frac{n}{n_c}\frac{\partial}{\partial r}\left(\frac{\epsilon_0\langle E^2\rangle}{2}\right)$$

$$\frac{1}{n}\frac{\partial n}{\partial r} - \frac{\epsilon_0}{2n_cKT}\frac{\partial}{\partial r}\langle E^2\rangle \qquad \ln n = -\frac{\epsilon_0\langle E^2\rangle}{2n_cKT} + \ln n_0$$

$$n = n_0\, e^{-\epsilon_0\langle E^2\rangle/2n_cKT}$$

At $r = 0$,

$$n_{\min} = n_0\, e^{-\epsilon_0\langle E^2\rangle_{\max}/2n_cKT} = n_0\, e^{-\alpha}$$

$$\therefore\ \alpha = \frac{\epsilon_0\langle E^2\rangle_{\max}}{2n_cKT}$$

8-9.

$$k_0 = 2\pi/\lambda_0 = 2\pi/1.06 \times 10^{-6} = 5.93 \times 10^6\ \text{m}^{-1}$$

$$k_i \approx 2k_0 = 1.19 \times 10^7\ \text{m}^{-1}$$

$$v_s = \left(\frac{KT_e + 3KT_i}{M}\right)^{1/2} = \left[\frac{(10^3)(1.6 \times 10^{-19})}{(2)(1.67 \times 10^{-27})}\right]^{1/2}\left(1 + \frac{3}{\theta}\right)^{1/2}$$

$$\omega_i = \Delta\omega = k_i v_s = (1.19 \times 10^7)(2.19 \times 10^5)\left(1 + \frac{3}{\theta}\right)^{1/2}$$

$$= 2.61 \times 10^{12}\left(1 + \frac{3}{\theta}\right)^{1/2}$$

$$\frac{\Delta\omega}{\omega_0} = -\frac{\Delta\lambda}{\lambda_0}\ \therefore\ \Delta\omega = -\frac{\omega_0}{\lambda_0}\Delta\lambda = -\frac{2\pi c}{\lambda_0^2}\Delta\lambda$$

Thus,

$$c_1 = \frac{ikF_{\mathrm{NL}}}{M}\frac{1}{\langle E_0E_2\rangle} = \frac{ik}{M}\left(\frac{-\omega_p^2}{\omega_0\omega_2}ik\epsilon_0\right) = \frac{\omega_p^2}{\omega_0\omega_2}\frac{k^2\epsilon_0}{M}$$

8-14. The upper sideband has $\hbar\omega_2 = \hbar\omega_0 + \hbar\omega_1$, so that the outgoing photon has *more* energy than the original photon $\hbar\omega_0$. The lower sideband would be expected to be more favorable energetically, since it is an exothermic reaction, with $\hbar\omega_2 = \hbar\omega_0 - \hbar\omega_1$.

8-18. $U(\xi - c\tau) = 3c\,\mathrm{sech}^2\,[(c/2)^{1/2}(\xi - c\tau)]$, where $\xi = \delta^{1/2}(x' - t')$, $\tau = \delta^{3/2}t'$, $x' = x/\lambda_{\mathrm{D}}$, $t' = \Omega_p t$, $\delta = \mathcal{M} - 1$

$$\zeta = \xi - ct = \delta^{1/2}\left(\frac{x - v_s t}{\lambda_{\mathrm{D}}} - \delta c\frac{v_s}{\lambda_{\mathrm{D}}}t\right)$$

since $\lambda_{\mathrm{D}}\Omega_p = v_s$

$$\zeta = \frac{\delta^{1/2}}{\lambda_{\mathrm{D}}}[x - (1 + \delta c)v_s t]$$

The soliton has a peak at $\zeta = 0$. The velocity of the peak is $dx/dt = (1 + \delta c)v_s$. By definition,

$$\frac{dx}{dt} = \mathcal{M}v_s = (1 + \delta)v_s$$

$$\therefore\ c = 1 \qquad \therefore\ U_{\max} = 3c = 3$$

From Eq. [8-111],

$$x_{\max} \equiv \frac{e\phi_{\max}}{KT_e} \approx \delta x_{1\max} = \delta U_{\max}$$

$$\therefore\ \delta = \frac{e}{KT_e}\frac{\phi_{\max}}{U_{\max}} = \frac{12}{10}\frac{1}{3} = 0.4$$

$$v_\phi = (1 + \delta)v_s = 1.4v_s$$

$$v_s = \left(\frac{KT_e}{M}\right)^{1/2} = \left[\frac{(10)(1.6 \times 10^{-19})}{1.67 \times 10^{-27}}\right] = 3.10 \times 10^4$$

$$v_\phi = \underline{4.33 \times 10^4\ \mathrm{m/sec}}$$

At half maximum, $\mathrm{sech}^2\, a = \frac{1}{2}\ \therefore\ a = 0.8814 = \sqrt{\tfrac{1}{2}}\zeta\ \therefore\ \zeta = 1.25 = \delta^{1/2}x/\lambda_{\mathrm{D}}$ at $t = 0$, say.

$$\delta^{1/2} = \sqrt{0.4} = 0.632$$

$$\lambda_{\mathrm{D}} = \left(\frac{\epsilon_0 KT_e}{n_0 e^2}\right)^{1/2} = 2.35 \times 10^{-4}\ \mathrm{m} = 0.235\ \mathrm{mm}$$

$$x = \frac{1.25\lambda_{\mathrm{D}}}{0.632} = 0.46\ \mathrm{mm} \qquad \mathrm{FWHM} = 2x = \underline{0.93\ \mathrm{mm}}$$

8-21.

$$|u| = 4A^{1/2}|\operatorname{sech} x| \quad \therefore \; |u|^2 = 16A|\operatorname{sech} x|^2$$

$$\delta n = \frac{1}{4}|u|^2\left(\frac{V^2}{\epsilon^2} - 1\right)^{-1} \approx -\frac{1}{4}|u|^2 = -4A|\operatorname{sech} x|^2$$

$$\overline{\delta n} = -4A\overline{|sech\, x|^2} \approx -2A$$

$$\frac{\delta\omega_p}{\omega_p} = \frac{1}{2}\frac{\delta n}{n} = -\frac{1}{2}(2A) = -A$$

$\therefore$ A is frequency shifted due to δn.

8-22. In real units,

$$u = \frac{v}{v_e} = 4A^{1/2}\operatorname{sech}\left[\left(\frac{2A}{3}\right)^{1/2}\left(\frac{x}{\lambda_D} - \frac{V}{v_e}\omega_p t\right)\right]\exp\left\{-i\left[\left(\frac{\omega_0}{\omega_p} + \frac{1}{6}\frac{V^2}{v_e} - A\right)\omega_p t\right.\right.$$

$$\left.\left. - \frac{V}{3v_e}\frac{x}{\lambda_D}\right\}\right.$$

$$v_e = \left(\frac{KT_e}{m}\right)^{1/2} = 5.93 \times 10^5 \text{ m/sec} \qquad \omega_p = \left(\frac{ne^2}{\epsilon_0 m}\right)^{1/2} = 1.78 \times 10^9 \frac{\text{rad}}{\text{sec}}$$

$$\lambda_D = \frac{v_e}{\omega_p} = 3.33 \times 10^{-4} \text{ m} \qquad k = \frac{(k\lambda_D)}{\lambda_D} = \frac{0.3}{\lambda_D} = 9.02 \times 10^2 \text{ m}^{-1}$$

$$u_{p\text{-}p} = 4A^{1/2} \qquad -i\omega m v = -eE = -e(-ik\phi) \;\therefore\; \phi = -\frac{m\omega v}{ek}$$

$$\phi_{p\text{-}p} \approx \frac{m\omega}{ek}4A^{1/2}v_e \qquad \omega \approx (\omega_p^2 + 3k^2 v_e^2)^{1/2} = 2.01 \times 10^9$$

$$A^{1/2} = \frac{ke\phi_{p\text{-}p}}{4m\omega v_e} = \frac{k}{4\omega}\frac{e\phi_{p\text{-}p}}{KT_e}\frac{KT_e}{m}\frac{1}{v_e} = \frac{kv_e}{4\omega}\frac{e\phi_{p\text{-}p}}{KT_e}$$

$$= \frac{kv_e}{4\omega}\frac{3.2}{2} = 0.106$$

$$A = 1.13 \times 10^{-2}$$

(a)

$$\operatorname{sech} X = \frac{1}{2} \qquad X = 1.315 = \left(\frac{2A}{3}\right)^{1/2}\frac{x}{\lambda_D}$$

$$x = \left(\frac{3}{2}\right)^{1/2}\frac{(1.315)(3.33 \times 10^{-4})}{0.106} = 5.04 \times 10^{-3}$$

$$\text{FWHM} = 2x = 1.01 \times 10^{-2} = 10.1 \text{ mm}$$

(b)

$$N = \frac{1.01 \times 10^{-2}}{2\pi/k} = 1.45$$

From Problem 8-6(a):

$$I_0 = c\epsilon_0\langle E^2\rangle = c\epsilon_0\frac{m^2\omega_0^2}{e^2}\langle v_0^2\rangle$$

$$I_0 = (3\times 10^8)(8.854\times 10^{-12})\frac{(0.91\times 10^{-30})^2(1.78\times 10^{14})^2(1.96\times 10^7)}{(1.6\times 10^{-19})^2}$$

$$= 5.34\times 10^{10}\frac{\mathrm{W}}{\mathrm{m}^2} = 5.34\times 10^6\frac{\mathrm{W}}{\mathrm{cm}^2}$$

8-11. $$(\omega_s^2 + 2i\gamma\omega_s - \omega_1^2)[(\omega_s + i\gamma - \omega_0)^2 - \omega_2^2] = \tfrac{1}{4}c_1c_2\bar{E}_0^2.$$

If $\omega_s^2 = \omega_1^2$, $(\omega_s - \omega_0)^2 = \omega_2^2$, and $\gamma/\omega_s \ll 1$, then

$$(2i\gamma\omega_s)[2i\gamma(\omega_s - \omega_0)] = \tfrac{1}{4}c_1c_2\bar{E}_0^2 = 4\gamma^2\omega_s\omega_2$$

From Problem 8-10,

$$c_1c_2 = \frac{\epsilon_0 k_1^2\omega_p^4}{n_0\omega_0^2 M} = \frac{k_1^2\omega_p^2 e^2}{\omega_0^2 mM}$$

$$\gamma^2 = \frac{k_1^2\omega_p^2e^2\bar{E}_0^2}{16\omega_s\omega_2\omega_0^2 mM} = \frac{k_1^2\omega_p^2\bar{v}_0^2 m}{16\omega_s\omega_2 M} \approx \frac{(2k_0)^2\Omega_p^2\bar{v}_0^2}{16\omega_0\omega_s}$$

$$= \frac{\omega_0^2\Omega_p^2\bar{v}_0^2}{4c^2\omega_0\omega_s} \quad \therefore\ \gamma = \frac{\bar{v}_0}{2}\left(\frac{\omega_0}{\omega_s}\right)^{1/2}\Omega_p$$

8–13. (a)

$$Mn_0\frac{\partial v}{\partial t} = en_0E - \gamma_i KT_i\nabla n - Mn_0\nu v + F_{\mathrm{NL}}$$

$$Mn_0(-i\omega + \nu)v = en_0(-ik\phi) - \gamma_i KT_i ikn_1 + F_{\mathrm{NL}}$$

with $e\phi/KT_e = n_1/n_0$, this becomes

$$(\omega + i\nu)v = kv_s^2\frac{n_1}{n_0} + \frac{iF_{\mathrm{NL}}}{Mn_0}$$

Continuity:

$$-i\omega n_1 + ikn_0v = -i\omega n_1 + ikn_0(\omega + i\nu)^{-1}\left[kv_s^2\frac{n_1}{n_0} + \frac{iF_{\mathrm{NL}}}{Mn_0}\right] = 0$$

$$(\omega^2 + i\nu\omega - k^2v_s^2)n_1 = ikF_{\mathrm{NL}}/M$$

When $F_{\mathrm{NL}} = 0$,

$$\omega^2\left(1 + i\frac{\nu}{\omega}\right) = k^2v_s^2 \quad \therefore\ \omega \approx kv_s\left(1 - \frac{1}{2}i\frac{\nu}{\omega}\right) = kv_s - \frac{i}{2}\nu$$

Hence $-\mathrm{Im}\,\omega \equiv \Gamma = \nu/2$. So $(\omega^2 + 2i\Gamma\omega - k^2v_s^2)n_1 = ikF_{\mathrm{NL}}/M$

(b)

$$F_{\mathrm{NL}} = -\frac{\omega_p^2}{\omega_0\omega_2}\nabla\epsilon_0\langle E_0E_2\rangle = -\frac{\omega_p^2}{\omega_0\omega_2}ik\epsilon_0\langle E_0E_2\rangle$$

(c)

$$\delta\omega = A\omega_p = (1.13 \times 10^{-2})(1.78 \times 10^9) = 2 \times 10^7 \text{ rad/sec}$$

$$\delta f = \delta\omega/2\pi = 3.2 \times 10^6 = 3.2 \text{ MHz}$$

8-23.

$$3v_e^2 = \frac{(3)(3)(1.6 \times 10^{-19})}{0.91 \times 10^{-30}} = 1.58 \times 10^{12} \text{ m}^2/\text{sec}^2$$

$$\omega_p^2(\text{out}) = \frac{(10^{16})(1.6 \times 10^{-19})^2}{(8.824 \times 10^{-12})(0.91 \times 10^{-30})} = 3.18 \times 10^{19} \frac{\text{rad}^2}{\text{sec}^2}$$

$$\omega_p^2(\text{in}) = 0.4\omega_p^2(\text{out})$$

$$k_{\max}^2 = \frac{\omega_p^2(\text{out}) - \omega_p^2(\text{in})}{3v_e^2} = \frac{3.18 \times 10^{19}}{1.58 \times 10^{12}}(1 - 0.4)$$

$$= 1.21 \times 10^7 \text{ m}^{-2}$$

$$\lambda_{\min} = \frac{2\pi}{k_{\max}} = 1.81 \times 10^{-3} \text{ m} = \underline{1.81 \text{ mm}}$$